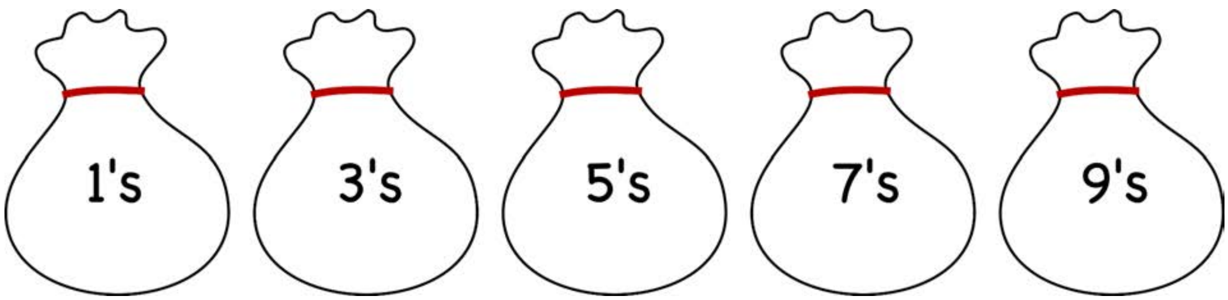
Puzzle of the Week
Adding Ten Numbers

THE CHALLENGE: You have five big bags of coins. Each bag has only one kind of coin. The bags have coins worth 1, 3, 5, 7, and 9. If you can, find ten coins that add up to 43. If you can't, describe why it is impossible.



EXPLORATION: What are all the possible numbers you can make with ten coins from these bags?

Puzzle of the Week

Adding Ten Numbers – Notes

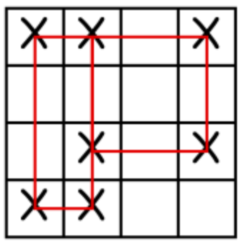
THE CHALLENGE: This is impossible. All the numbers to choose from are odd. Odd numbers added together an even number (ten) times will always produce an even number, no matter which odd numbers are chosen.

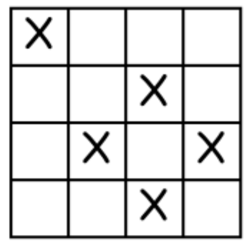
EXPLORATION: Every even number from 10 to 90 can be produced. Start with 10 = 10 x 1. To produce all the even numbers up to 90, replace one number each time with a number that is two larger than it.

Puzzle of the Week

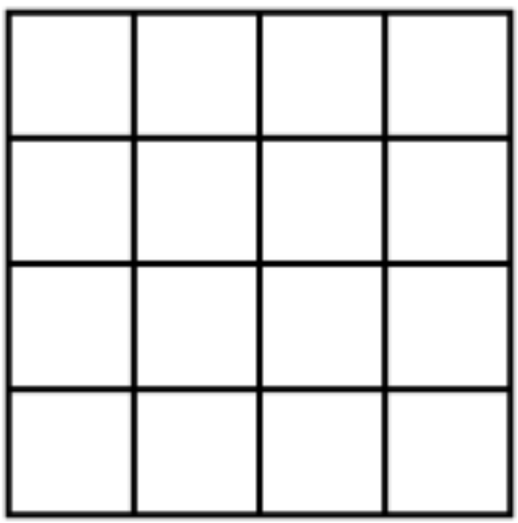
Avoiding Rectangles – 2

The X's in a grid can become the corners of rectangles with horizontal and vertical sides. The X's in this first grid form two rectangles. However, the goal is to avoid creating rectangles. The X's in the second grid are placed to avoid forming any rectangles.





THE CHALLENGE: Place as many X's as you can in this 4 by 4 grid and avoid creating any rectangles.



EXPLORATION: What is the most you can have for other sizes of grids? Do you see any patterns in your answers?

Puzzle of the Week

Avoiding Rectangles – 2 – Notes

THE CHALLENGE: To solve this in general is a very hard, unsolved problem in mathematics. However, that should not keep a student from playing around with it and learning a lot of interesting things along the way.

Suppose the grid has m rows and n columns. In Avoiding Rectangles – 1, we found the following results for the best answers.

1 and 2 rows: We had m + n - 1 X's

3 rows: We had m + n X's using a starting diagram for a 3 by 3 grid that looked like the following diagram. Using that diagram, if we had 3 rows with more columns, we can just add a single X in each new column.

X		X
X	X	
	X	X

The 3 by 3 case required a different look from the 1 and 2 row strategy, and it is tempting to think that the examples with 4 or more rows will also require some new thinking. To state the obvious, we want to create a collection of columns that never share a pair of rows with X's in them.

One strategy is to fill in everything in the leftmost column except the bottom square, and then the rest of the columns have one X in the bottom row and one X in another row. This will always work and produces (m - 1) + 2 x (m - 1) = 3 x (m - 1) X's for an m by m square grid. This method for filling in the square can then be extended (by adding a single X in each new column) to anything with m rows to give 3 x (m - 1) + (n - m) = n + 2m - 3 X's, when m < n. This formula works for all the answers for any size grid (with more than 1 row) so far.

X	X		
X		X	
X			X
	X	X	X

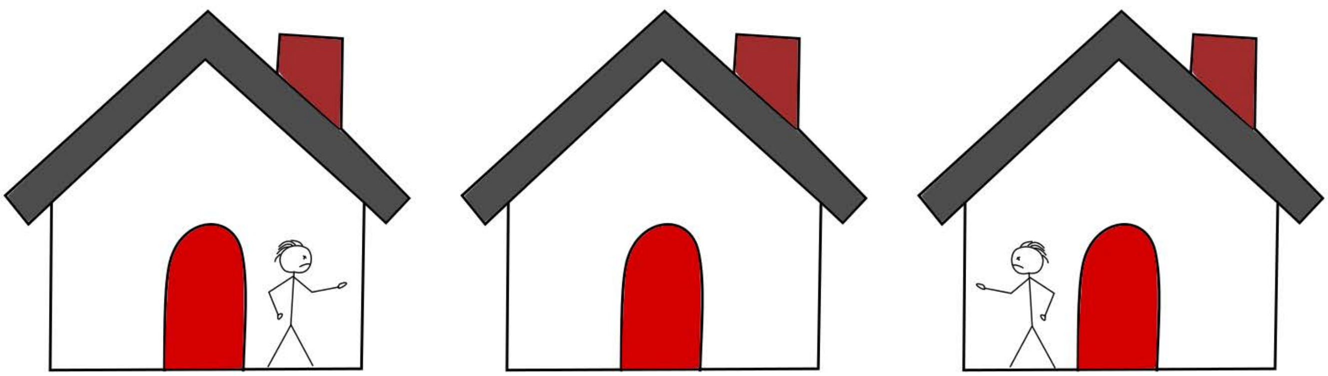
EXPLORATION: Looking for the best answers online shows you can do 12 X's for the 5 by 5 case (which you can get using the pattern), and then 16 X's for the 6 by 6 case (shown here).

X	X			X	
X		X			X
X			X		
	X		X		X
	X	X			
		X	X	X	

After that, the numbers starting at 7 by 7 in order are: 21, 24, 29, 34, 39, 45, 52, 56, 61, 67, 74, 81, 88, 96, 105, 108, 115, and 122. Mathematicians have done a lot of interesting explorations for how these work, and your mathematicians can have a lot of fun with it too!

Puzzle of the Week
Bad Neighbors

THE CHALLENGE: There are a group of people who cannot live next door to each other in houses along a street. If there are 10 houses, in how many ways can some of these people live in these houses so that no two neighboring houses have these people in them? Count having all the houses empty as one of the possibilities.



EXPLORATION: How many ways are there if there are 15 houses? Find a pattern that will help you quickly calculate how many different ways for larger numbers of houses.

Puzzle of the Week

Bad Neighbors – Notes

THE CHALLENGE & EXPLORATION: As with so many puzzles, the best way to learn about this is by doing simple examples looking for patterns. Use E for empty and B for a bad neighbor.

- 1 house - 2 ways: E or B
- 2 houses - 3 ways: EE, BE, EB
- 3 houses - 5 ways: EEE, BEE, EBE, EEB, BEB
- 4 houses - 8 ways: EEEE, BEEE, EBEE, EEBE, EEEB, BEBE, BEEB, EBEB

It looks like they are the Fibonacci Numbers. Let's find a good reason why that should be.

To find the number of ways to fill n houses, start with either an E or a B.

- If you start with an E, then the next n-1 houses can be filled ignoring that first house.
- If you start with a B, then the next house must be an E. After those two, the next n-2 houses can be filled ignoring the first two houses.

Therefore, the number of ways of filling n houses equals the number of ways of filling n-1 houses plus the number of ways of filling n-2 houses. This is exactly the rule for calculating the Fibonacci Numbers. Because we start with 2 and 3, the rule will force the rest of the numbers to be the same after that.

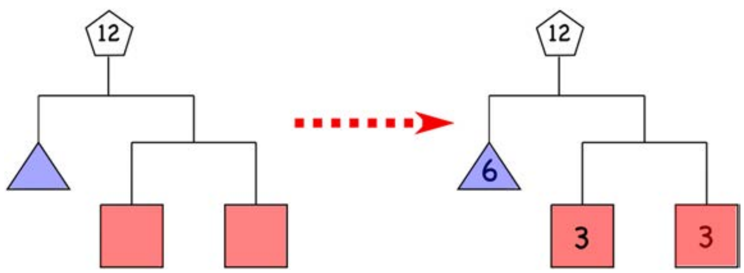
The numbers will be: 2, 3, 5, 8, 13, 21, 34, 55, 89, 144 for the first ten.

Puzzle of the Week

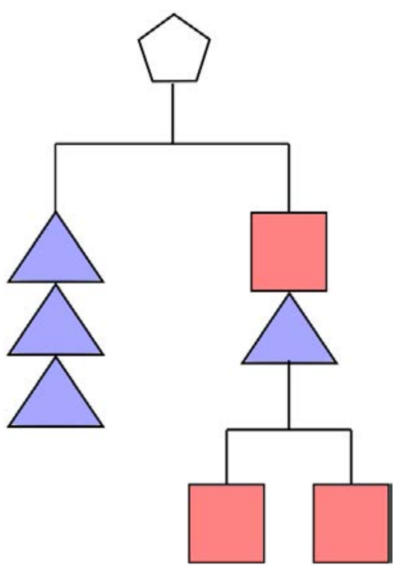
Balance Beam – 1

To balance, weights must be the same on opposite sides of a horizontal balance beam. The total weight is given above the balance beam.

In a given puzzle, figures of the same shape must have the same weight. However, it is allowed for different shapes to have the same weight.



THE CHALLENGE: The squares each have weight 2. Find the weight of each of the triangles and the total weight of all the figures.



EXPLORATION: Create balance beams for others to solve. Make sure there is enough information that they can be figured out.

Puzzle of the Week

Balance Beam – 1 - Notes

THE CHALLENGE: Replace the squares with 2's. This means that the three triangles on the left side balance with a triangle plus 6 (three 2's) on the right. For these two sides to be equal, two triangles must balance with the 6. So, each triangle has a weight of 3.

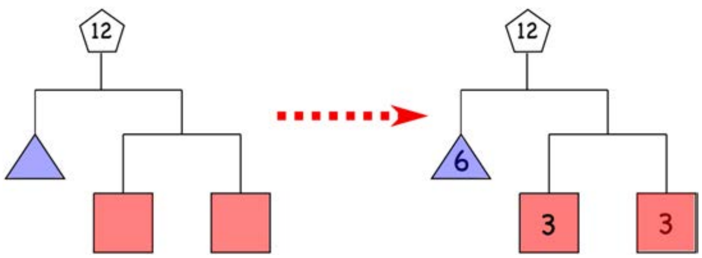
Replacing the triangles with 3's means that there are a total of 9 on each side, which gives a grand total of 18 for the whole balance beam.

Puzzle of the Week

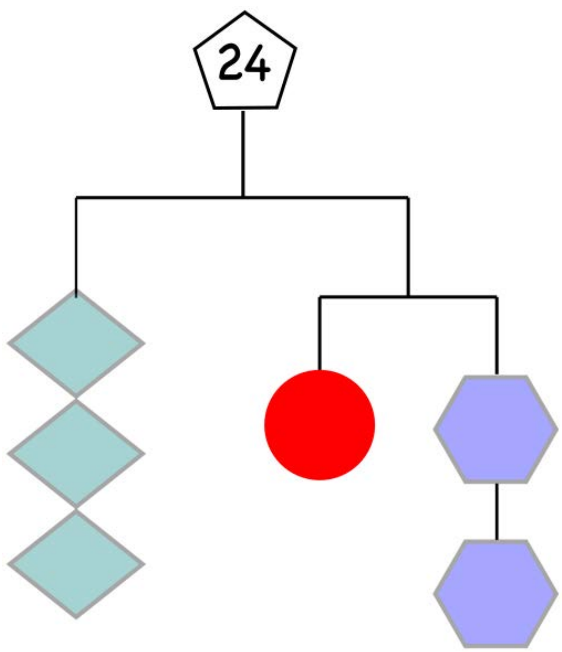
Balance Beam – 2

To balance, weights must be the same on opposite sides of a horizontal balance beam. The total weight is given above the balance beam.

For these puzzles, figures of the same shape must have the same weight. However, it is allowed for different shapes to have the same weight.



THE CHALLENGE: Find the weight of each of the diamonds, hexagons, and circles.



EXPLORATION: Create balance beams with at least three shapes for others to solve. Make sure there is enough information that they can be figured out.

Puzzle of the Week

Balance Beam – 2 – Notes

THE CHALLENGE: Because the two sides of the big beam are equal and add up to 24, each side must be 12.

The three diamonds on the left side add up to 12, so each diamond is 4.

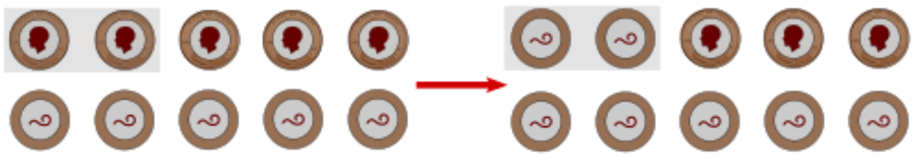
The two sides of the smaller beam are equal and add up to 12. This means the red circle is 6 and the two hexagons add up to 6. Consequently, each hexagon is 3.

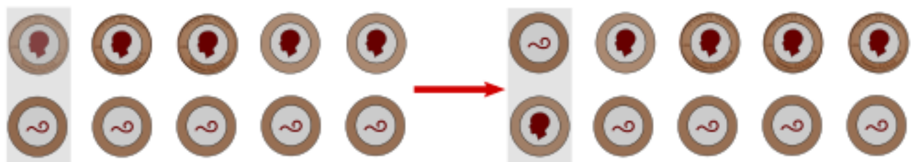
To summarize: a diamond is 4, a red circle is 6, and a hexagon is 3.

Puzzle of the Week

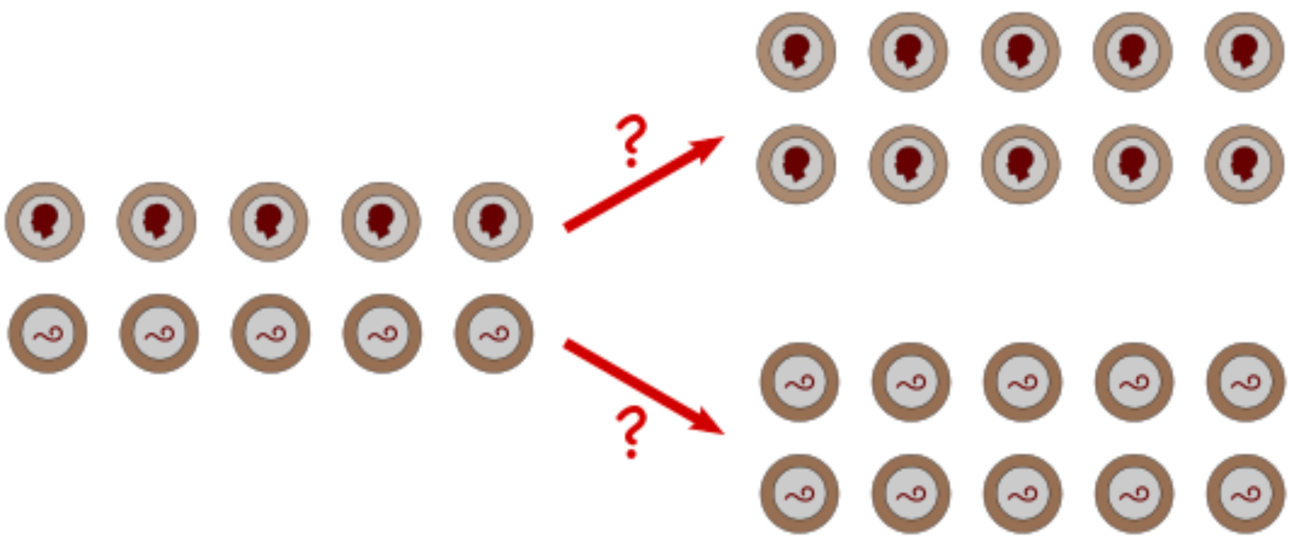
Coin Flipping – 3

Ten coins are set up as shown on the left side. The first five coins are heads and the second five are tails. During one move, you are allowed to flip over any two coins.





THE CHALLENGE: Take the original configuration of coins and, through a series of double-flip moves, make all ten coins heads or all ten tails. Show how this can be done or describe why it is impossible.



Puzzle of the Week
Coin Flipping – 3 – Notes

THE CHALLENGE: Consider the number of heads before any move involving two flips. There are three possibilities when flipping two coins.

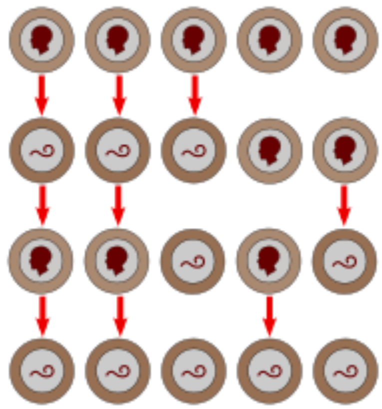
- **The two coins were both heads up to start.** After the flips, these coins will be heads down, and the number of heads up coins will be reduced by two.
- **The two coins were both heads down to start.** After the flips, these coins will be heads up, and the number of heads up coins will be increased by two.
- **One coin was heads up and the other was heads down.** After the flips, the coins will be heads down and heads up, so there will be no change in the number of heads up coins.

Consequently, the number of heads up coins will either increase by two, stay the same, or decrease by two. The number of heads up coins starts at five, and will always change by an even amount, so it must remain an odd number no matter what we do. Therefore, this number can never become ten or zero, and so it is impossible to have all the coins end up in the same orientation.

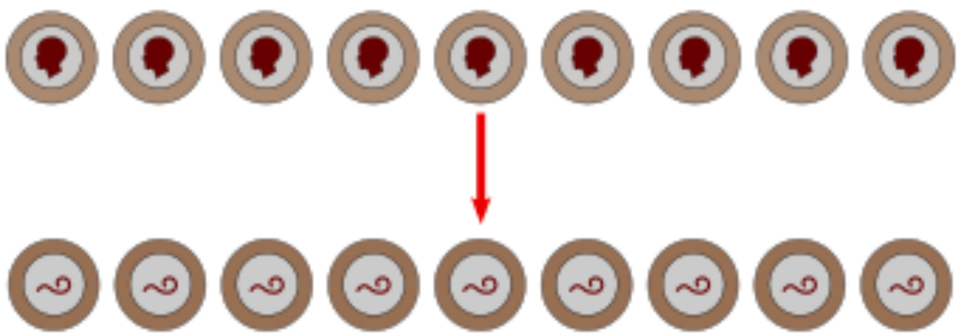
Puzzle of the Week

Coin Flipping – 4

Choosing 3 coins during each turn, you can start with 5 heads and end with 5 tails.



THE CHALLENGE: Choosing 5 coins during each turn, start with 9 heads and end with 9 tails. What's the fewest number of turns you need to use?



EXPLORATION: What happens if 9 and 5 are replaced by two other numbers? Can you predict when it's possible? Can you predict what the fewest number of turns will be?

Puzzle of the Week

Coin Flipping – 4 – Notes

THE CHALLENGE & EXPLORATION: Suppose there are n coins and k flips being done for each turn. To go from all heads to all tails, there needs to be n flips plus possibly some coins flipped two extra times to make things come out evenly. The total number of flips will be n + (2 x extras).

To come out evenly, k must evenly divide n + (2 x extras). In the example in the introduction, which has n = 5 and k = 3. 3 evenly divides 5 + 2 x 2 = 9, so the first time it can possibly work is after 3 turns. Note that there are two coins that are flipped three times instead of just once.

In the Challenge, n = 9 and k = 5. The first time 5 evenly divides 9 + (2 x extras) is for 9 + 2 x 3 = 15. It will take three turns, and there will be three coins that are flipped an extra two times.

If n is odd and k is even, it will be impossible. That's because an even number will never evenly divide an odd number plus (2 x extras), which is an odd number.

Note that some care needs to be exercised in simply assuming that just because k even divides n + (2 x extras) that it will work. For example, let n = 4 and k = 3. 3 evenly divides 4 + 2 x 1 = 6. However, it is not possible to do this in two turns. It will take four turns where 3 evenly divides 4 + 2 x 4 = 12.

Puzzle of the Week

Combining Digits – 1 2 4 8

Here are some ways to get 0 and 1 using 1, 2, 4, and 8.

0 = 8 - 1 * 2 * 4

0 = 8 * 1 - 2 * 4

1 = 8 - 2 * 4 + 1

1 = 8 - 4 - 2 - 1

THE CHALLENGE: How many numbers can you get using each of the numbers 1, 2, 4, and 8 in any order, using addition, subtraction, and multiplication?

EXPLORATION: What happens with other groups of four numbers? What happens if you use the five numbers: 1, 2, 4, 8, and 16?

Puzzle of the Week

Combining Digits – 1 2 4 8 – Notes

THE CHALLENGE: Here are some solutions from 0 to 26. Of course, there are many more. Have fun comparing different people's solutions!

0 = 8 - 1 * 2 * 4
1 = 8 - 4 - 2 - 1
2 = 8 - 4 - 2 * 1
3 = 8 - 4 - 2 + 1
4 = 8 - 4 * (2 - 1)
5 = 8 - 4 + 2 - 1
6 = 8 - 4 + (2 * 1)
7 = 8 - 4 + 2 + 1
8 = 8 * (4 - 2 - 1)
9 = 8 + 4 - 2 - 1
10 = 8 + 4 - (2 * 1)
11 = 8 + 4 - 2 + 1
12 = 8 + 4 * (2 - 1)
13 = 8 + 4 + 2 - 1
14 = 8 + 4 + (2 * 1)
15 = 8 + 4 + 2 + 1
16 = 8 * (4 - (2 * 1))
17 = 8 * (4 - 2) + 1
18 = (8 + 1) * (4 - 2)
19 = 8 * 2 + 4 - 1
20 = 8 * 2 + 4 * 1
21 = 8 * 2 + 4 + 1
22 = 8 * (4 - 1) - 2
23 = (8 - 2) * 4 - 1
24 = (8 - 2) * 4 * 1
25 = (8 - 2) * 4 + 1
26 = 8 * (4 - 1) + 2

Puzzle of the Week

Combining Digits – Easy as 1 2 3 4

Here are some ways to get 0 and 1 using 1, 2, 3, and 4.

0 = 1 + 4 - 2 - 3

0 = (3 - 1 - 2) * 4

1 = (2 - 1) * (4 - 3)

1 = 4 - 3 * (2 - 1)

THE CHALLENGE: How many numbers can you get using each of the numbers 1, 2, 3, and 4 in any order, using addition, subtraction, and multiplication?

EXPLORATION: How many more numbers can you make if you are also allowed to make two-digit numbers with the digits? For example, 26 = 24 + 3 - 1.

Puzzle of the Week

Combining Digits – 1 2 3 4 – Notes

THE CHALLENGE: Here are some solutions, with one missing, from 0 to 21. Of course, there are many more. Have fun comparing different people's solutions!

0 = 1 + 4 - 2 - 3
1 = 4 - 3 * (2 - 1)
2 = (2 - 1) + (4 - 3)
3 = 4 – (3 – 2 * 1)
4 = 4 * (3 – 2*1)
5 = 4 + (3 – 2*1)
6 = 4 + 3 – 2 + 1
7 = 4 + 3 * (2 – 1)
8 = 4 * (3 – (2 – 1))
9 = 3 * (4 – (2 – 1))
10 = 1 + 2 + 3 + 4
11 = 3 * 4 – (2 - 1)
12 = 3 * 4 * (2 – 1)
13 = 3 * 4 + (2 – 1)
14 = 2 * (3 + 4) * 1
15 = 2 * (3 + 4) + 1
16 = 2 * (1 + 3 + 4)
17 =
18 = 4 * (3 + 1) + 2
19 = 4 * (2 + 3) – 1
20 = 4 * (2 + 3) * 1
21 = 4 * (2 + 3) + 1

EXPLORATION: Here are solutions up to 37 making use of two-digit numbers.

18 = 23 - 4 - 1
19 = 23 - 4 * 1
20 = 24 - 3 - 1
21 = 24 - 3 * 1
22 = 2 * 13 - 4
23 = 4 * 3 * 2 - 1
24 = 4 * 3 * 2 * 1
25 = 2 * 14 - 3
26 = 24 + 3 - 1.
27 = 23 + 1 * 4
28 = 23 + 4 + 1
29 = 32 - 4 + 1
30 = (4 + 1) * 3 * 2
31 = 34 - 2 - 1
32 = 34 - 2 * 1
33 = 34 - 2 + 1
34 = 34 * (2 - 1)
35 = 34 + 2 - 1
36 = 34 + 2 * 1
37 = 34 + 2 + 1

Puzzle of the Week

Combining Digits – Five 2’s

Here are some ways to get 0 and 1 using five 2’s.

$$0 = (22 - 22) \times 2$$

$$0 = (2/2 - 2/2) \times 2$$

$$1 = 2 - (2/2 \times 2/2)$$

$$1 = (2/2 + 2/2) / 2$$

THE CHALLENGE: How many numbers can you get using five 2’s using addition, subtraction, multiplication, division and creating double-digit numbers?

2 2 2 2 2

EXPLORATION: Play with other groups of identical numbers. Are some more interesting than others?

Puzzle of the Week

Combining Digits – Five 2's – Notes

THE CHALLENGE: Here are some solutions from 0 to 16. Of course, there are many more. Have fun comparing different people's solutions!

0 = (2 - 2) / (2 x 2 x 2)
1 = (2 / 2 + 2 / 2) / 2
2 = 2 + (2 - 2) x (2 + 2)
3 = 2 + (2/2 x 2/2)
4 = (2 + 2 + 2 + 2) / 2
5 = 2 + (2 + 2 + 2) / 2
6 = (2 x (2 + 2 + 2)) / 2
7 = 2 + 2 + 2 + 2/2
8 = 2 x 2 x 2 x 2/2
9 = 2 x 2 x 2 + 2/2
10 = 22/2 - 2/2
11 = 22/2 x 2/2
12 = 22/2 + 2/2
13 = (22 + 2 + 2) / 2
14 = 22 - (2 x 2 x 2)
15 = 22/2 + 2 + 2
16 = 22 - 2 - 2 - 2

Puzzle of the Week

Combining Digits – Four 4's

Here are some ways to get 0 and 1 using four 4's.

0 = 4 - 4 + 4 -4

0 = 44 - 44

1 = 4 / 4 * 4 / 4

1 = 4 / 4 + 4 - 4

1 = 44 / 44

THE CHALLENGE: How many numbers can you get using four 4's using addition, subtraction, multiplication, division and creating double-digit numbers?

Puzzle of the Week

Combining Digits – Four 4's – Notes

THE CHALLENGE: Here are some solutions, with a few missing, from 0 to 16. Of course, there are many more. Have fun comparing different people's solutions!

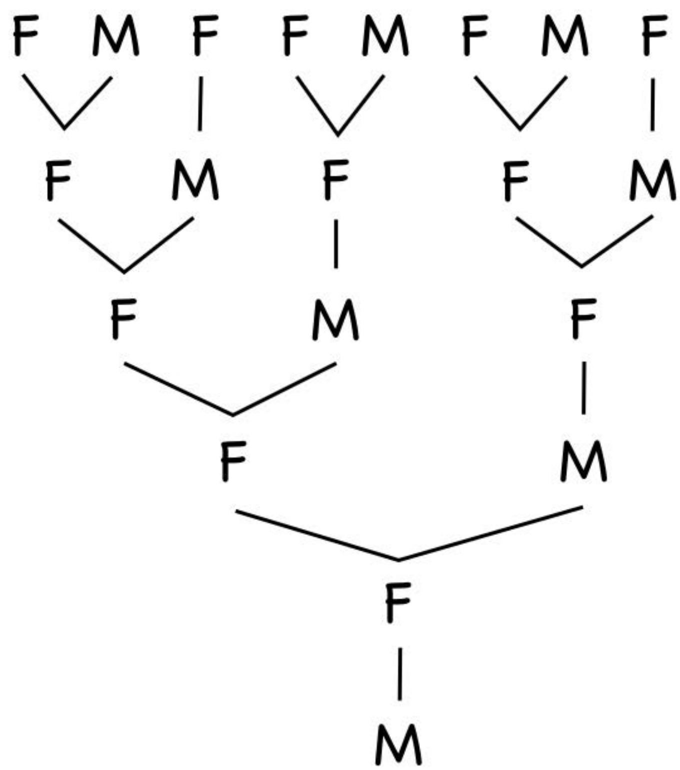
0 = 4 - 4 + 4 -4
1 = 4 / 4 + 4 - 4
2 = 4 / 4 + 4 / 4
3 = (4 + 4 + 4) / 4
4 = 4 + ((4 – 4) * 4)
5 = (4 * 4 + 4) / 4
6 = 4 + (4 + 4) / 4
7 = 4 + 4 – (4 / 4)
8 = (4 + 4) + 4 – 4
9 = 4 + 4 + (4 / 4)
10 = (44 – 4) / 4
11 =
12 = 4 * (4 – 4 / 4)
13 =
14 =
15 = 4 * 4 – (4 / 4)
16 = 4 * 4 + 4 – 4

Puzzle of the Week

Counting the Ancestors of Bees

A female bee's egg can have one of two things happen. If a male bee is not involved, then the egg will turn into a male bee. If a male bee is involved, the egg will turn into a female bee.

The ancestry of a typical male bee starts with a mother and no father - it has 1 ancestor 1 generation back. Its mother has a mother and father, so it has two grandparents - it has 2 ancestors 2 generations back. And so on. Here is the start of its ancestral tree.



THE CHALLENGE: How many ancestors will a male bee have 10 generations back? How about 20 generations back?

EXPLORATION: Find a pattern that will help you make these calculations more easily. What does the ancestry of a female bee look like? What simplifications are we making in doing all this counting?

Puzzle of the Week

Counting the Ancestors of Bees – Notes

THE CHALLENGE & EXPLORATION: These numbers are the Fibonacci Numbers. After doing a few generations of the bee's ancestors, the numbers are: 1 (the male bee), 1 (the mother), 2 (the mother and father), 3, 5, 8, 13, 21, 34, 55, and 89.

The key to making this easy to calculate is the following. For a given generation, everyone of the bees will have a mother. The number of fathers is the number of mothers in that generation, and that is exactly the number of bees in the generation before the given generation. In other words, the number of bees in the next generation is equal to the number of bees in this generation plus the number of bees in the previous generation.

This is exactly the rule for finding Fibonacci Numbers. The next Fibonacci Number is always the sum of the previous two numbers.

Our sequence is 1, 1, 2, 3, 5, <u>8</u>, 13, 21, 34, 55, <u>89</u>, 144, 233, 377, 610, <u>987</u>, 1597, 2584, 4181, 6765, and <u>10946</u>. I have underlined every fifth number, so the tenth number is 89 and the 20th number is 10946.

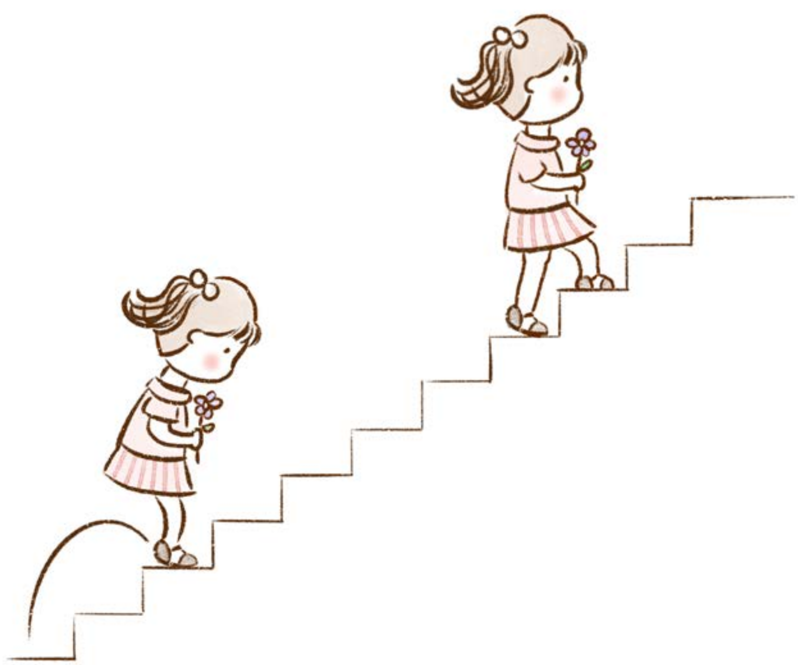
The numbers for the generations of a female bee will be exactly the same, only moved forward one generation.

The simplification we are making is that we assume that no bee is involved more than once as the ancestor of a given bee.

Puzzle of the Week
Counting Steps

When climbing steps, some people like to take the steps two at a time, at least part of the time. That leads to the general question of how many different ways there are of going up steps this way.

THE CHALLENGE: How many different ways are there of going up 10 steps where you are making some combination of single and double steps along the way (it may be all single steps or all double steps or some mixture)? How about 20 steps?



EXPLORATION: Do you see a pattern you recognize? Can you find a shortcut that will make it easier to count all the ways of doing this?

Puzzle of the Week

Counting Steps – Notes

THE CHALLENGE & EXPLORATION: This is another visit with Fibonacci Numbers. The first few counts of how many ways there are of doing this are: 1, 2, 3, 5, and 8. After doing the bee ancestry puzzle, and then seeing the first few numbers for this one, your students should be suspicious.

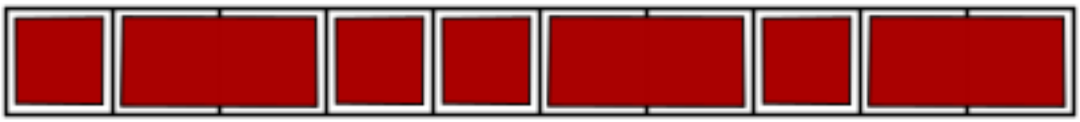
The tricky part is seeing where the Fibonacci rule comes into play. Let's look at going up ten steps as an example. You can divide the ways of doing ten steps into two buckets. In the first one is all the ways of going up nine steps and then taking a single step to get to the tenth. The other one is all the ways of going up eight steps and then take a double step to get to the tenth. Every possibility is counted this way, and nothing is counted twice.

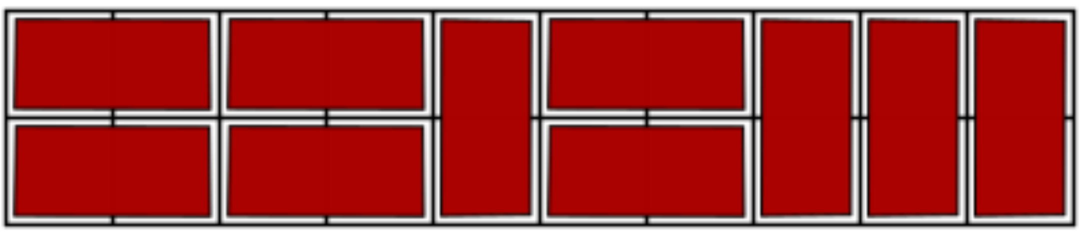
In general, the number of ways of going up n steps will be the number of ways of going up (n - 1) steps plus the number of ways of going up (n - 2) steps. That is exactly the Fibonacci rule.

Puzzle of the Week

Tiling Rectangles

You have two puzzles. For a 1 by n rectangle, how many ways can you fill it with a mixture of 1 by 1 squares and 1 by 2 rectangles? For a 2 by n rectangle, how many ways can you fill it with 1 by 2 rectangles?





THE CHALLENGE: Why do these two puzzles give the same answers? What are the answers when the rectangles being filled are 10 long? How about when they are 20 long?

EXPLORATION: How are these two problems similar to each other and earlier puzzles? How do things change in the 1 by n puzzle if we use 1 by 1 and 1 by 3 pieces instead? How do things change in the 1 by n puzzle if we use 1 by 1, 1 by 2, and 1 by 3 pieces?

Puzzle of the Week

Tiling Rectangles – Notes

THE CHALLENGE: Seeing how to apply earlier results to part or all of a new puzzle is a powerful skill.

These two puzzles are exactly the same in nature. Vertical pieces in the 2 x n rectangle correspond to 1 by 1 pieces in the 1 x n rectangle, and horizontal pieces in the 2 x n rectangle correspond to 1 by 2 pieces in the 1 x n rectangle. So, there really is no difference between the two puzzles.

Also, the 1 x n rectangle puzzle is exactly like the steps puzzle in "Fibonacci - 2." Going one step at a time is like putting in a 1 by 1 square, and going two steps at a time is like putting in a 1 by 2 rectangle. The two puzzles are the same, so the analysis and results are the same.

EXPLORATION: If we use 1 by 1 and 1 by 3 rectangles, a lot changes. Look at the first few values and consider how they are calculated. We get 1, 1, 2, 3, 4, 6, and 9 for the first few values. In general the next value is the sum of the current value and the one two steps before that. It is simple enough to calculate, but it is no longer the Fibonnaci sequence.

If we use 1 by 1, 1 by 2, and 1 by 3, things change even more dramatically. Now the first few values are given by 1, 2, 4, 7, 13, and 24. The next value in the sequence is the sum of the previous three values.

Sequences like this that define their next terms by a set formula involving previous terms are called recursive sequences. The Fibonacci Numbers are not the only recursive sequence, but they are probably the most famous.

Puzzle of the Week

Fill in the Blanks – 6

These sums, using the numbers from 1 to 6 once each, are not particularly close to 100.

$$\begin{array}{r} \boxed{3}\,\boxed{1} \\ \boxed{4}\,\boxed{2} \\ +\ \boxed{6}\,\boxed{5} \\ \hline 1\ \ 3\ \ 8 \end{array} \qquad \begin{array}{r} \boxed{1}\,\boxed{3} \\ \boxed{2}\,\boxed{5} \\ +\ \boxed{4}\,\boxed{6} \\ \hline 8\ \ 4 \end{array}$$

THE CHALLENGE: Use the numbers from 1 to 6 once each to make a sum as close to 100 as possible.

$$\begin{array}{r} \square\,\square \\ \square\,\square \\ +\ \square\,\square \\ \hline \end{array}$$

1 2 3 4 5 6

EXPLORATION: How does your answer change if you use the numbers from 1 to 7 or 1 to 8 instead, using each number no more than once.

Puzzle of the Week

Fill in the Blanks – 6 – Notes

THE CHALLENGE: A useful thing to notice is that it does not matter how we pair up the numbers in the ones column with the numbers in the tens column, so that is one less thing to think about.

To get close to 100, we want the tens column to add up to 8, 9, or 10. Because the sum of the numbers from 1 to 6 is 21, we can subtract the sum of the tens column from 21 to see what the sum of the ones column will be. In all there can be only three possible sums to consider:

- The tens column sums to 8 and the ones column sums to 13, which gives an overall sum of 93.
- The tens column sums to 9 and the ones column sums to 12, which gives an overall sum of 102.
- The tens column sums to 10 and the ones column sums to 11, which gives an overall sum of 111.

Of these 102 is the best. Any combination of digits in the tens column that sums to 9 will give the best possible answer. The things that work are 1 + 2 + 6, 1 + 3 + 5, and 2 + 3 + 4.

Here are some typical answers all of which sum to 102: 13 + 24 + 65, 16 + 34 + 52, and 21 + 35 + 46.

EXPLORATION: The analysis for the numbers from 1 to 7 is nearly the same as before. We still want the tens column to add up to 8, 9, or 10. The sum of the remaining digits will have some variability thanks to the wider range.

- The tens column sums to 8. The largest that 3 of the remaining 4 numbers can sum to is 17 (4 + 6 + 7), with overall sum 97.
- The tens column sums to 9. The smallest that 3 of the remaining 4 numbers can sum to is 12, with overall sum 102. The same answer as using the numbers from 1 to 6. The 7 is of no use here.
- The tens column sums to 10. We don't need to look at this as it will be too large.

To look at 1 to 8, we only need to look at the tens column adding up to 8 or 9. As before, if the tens column sums to 9, the best we can do will be 102. The question is, can we do better when the tens column sum is 8? If we use 1 + 3 + 4 in the tens column, then we can use 5 + 7 + 8 = 20 in the ones column. We can hit 100 exactly!

So, one best answer for 1 to 8 is 15 + 37 + 48 = 100. There are plenty more that you can create by moving around the ones column numbers.

Puzzle of the Week

Fill in the Blanks – 7

These sums, using the numbers from 1 to 9 once each, are not particularly close to 1000.

$$\begin{array}{r} \boxed{1}\,\boxed{2}\,\boxed{7} \\ \boxed{4}\,\boxed{3}\,\boxed{8} \\ +\ \boxed{6}\,\boxed{9}\,\boxed{5} \\ \hline 1\ 2\ 6\ 0 \end{array} \qquad \begin{array}{r} \boxed{1}\,\boxed{7}\,\boxed{3} \\ \boxed{2}\,\boxed{9}\,\boxed{8} \\ +\ \boxed{4}\,\boxed{6}\,\boxed{5} \\ \hline 9\ 3\ 6 \end{array}$$

THE CHALLENGE: Use the numbers from 1 to 9 once each to make a sum as close to 1000 as possible.

$$\begin{array}{r} \boxed{}\,\boxed{}\,\boxed{} \\ \boxed{}\,\boxed{}\,\boxed{} \\ +\ \boxed{}\,\boxed{}\,\boxed{} \\ \hline \end{array}$$

1 2 3 4 5 6 7 8 9

Puzzle of the Week

Fill in the Blanks – 7 – Notes

THE CHALLENGE: A useful thing to notice is that it does not matter how we pair up the numbers in the ones column, the tens column, and the hundreds column, so that is one less thing to think about.

Note that the sum of all the digits from 1 to 9 is 45. From a bit of number theory (think "casting out 9's"), because the sum of the digits is a multiple of 9, the resulting sum of all three three-digit numbers must be a multiple of 9. Therefore, the closest number we can hope for is 999, with 1008, and 990 coming in close behind that.

To get close to 1000, we want the hundreds column to add up to 8 or 9. So, we consider those two cases.

Case 1: The hundreds column sums to 8 (either as 1 + 2 + 5 or 1 + 3 + 4). In this case, we want the sum of the three two-digit numbers in the tens and ones column to be as close to 200 as possible. In that case, we want the sum of the tens digits to be 18, 19, or 20 - the corresponding sums of the ones digits will be 45 - (8 + 18) = 19, 45 - (8 + 19) = 18, and 45 - (8 + 20) = 17. The overall sums corresponding to these three scenarios are then 999, 1008, and 1017.

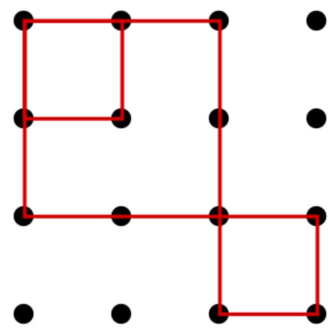
We can't do any better than 999. We want the hundreds digits to sum to 8, the tens digits to sum to 18, and the ones digits to sum to 19. Here are some sample answers: 152 + 368 + 479 = 999 and 194 + 237 + 568 = 999.

Case 2: The hundreds column sums to 9. In this case, we want the sum of the three two-digit numbers in the tens and ones column to be as close to 100 as possible. However, given that the hundreds digits add up to 9, the smallest three more numbers for the tens column can add up to is 21 - 9 = 12 (the 21 comes from the sum of the numbers from 1 to 6). So, this is hopeless.

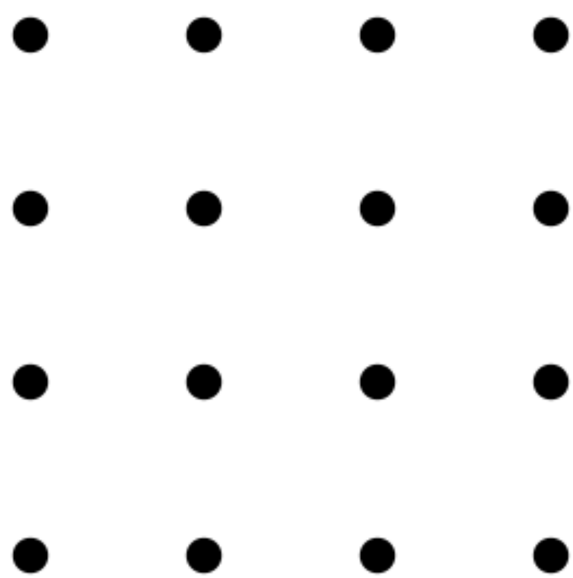
Puzzle of the Week

Finding Squares – 1

Drawn in red in this grid are two 1 by 1 squares and one 2 by 2 square with horizontal and vertical sides.



THE CHALLENGE: FInd the number of squares of all sizes with horizontal and vertical sides in this grid.



EXPLORATION: Play with even bigger grids. Can you discover a systematic way to count all the squares? How does your answer change, if at all, if the grid of points is a rectangle that is not a square?

Puzzle of the Week

Finding Squares – 1 – Notes

THE CHALLENGE: The possible squares have side lengths from 1 up to 4.

For each size of square, think of the upper left corner as the starting position. You can produce all squares of this size in the grid by shifting this starter square around. You have the choice of moving it to the right or down the remaining number of positions. These are independent choices, so you get <this number plus 1> squared of that size square.

For example, suppose you are counting 1 by 1 squares in this 4 by 4 grid. A 1 by 1 square in the upper left-hand corner can be shifted 1 or 2 to the right and 1 or 2 down. So there are 3 possible positions horizontally (counting the original position) and 3 possible positions vertically. Thus there are 3 x 3 of these 1 by 1 squares.

For the original 4 by 4 grid problem, this gives the following count:

- 1 by 1 squares: 3 x 3 = 9
- 2 by 2 squares: 2 x 2 = 4
- 3 by 3 squares: 1 x 1 = 1.

There are a total of 3 x 3 + 2 x 2 + 1 x 1 = 9 + 4 + 1 = 14 possible squares.

EXPLORATION: For any square grid, the total number of squares will be the sum of the square numbers less than the size of the grid. For example, a 7 by 7 grid will have 36 + 25 + 16 + 9 + 4 + 1 = 91 squares.

Rectangular grids are almost as easy to count. The smaller of the two dimensions of the rectangle will restrict the maximum size of the possible squares. Count them in a similar way as the square grid case. Rather than doing the general case, let's just do one illustrative example.

Suppose you are counting the squares in a 5 by 7 grid. Because there are only 5 rows, no square can be bigger than 4 by 4.

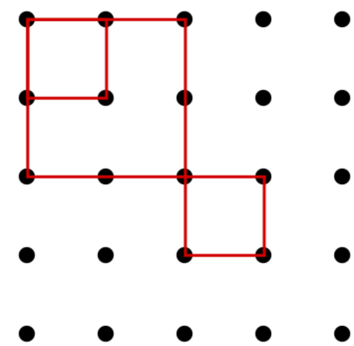
- 1 by 1 squares: 4 x 6 = 24
- 2 by 2 squares: 3 x 5 = 15
- 3 by 3 squares: 2 x 4 = 8
- 4 by 4 squares: 1 x 3 = 3

The total number of squares is 24 + 15 + 8 + 3 = 50.

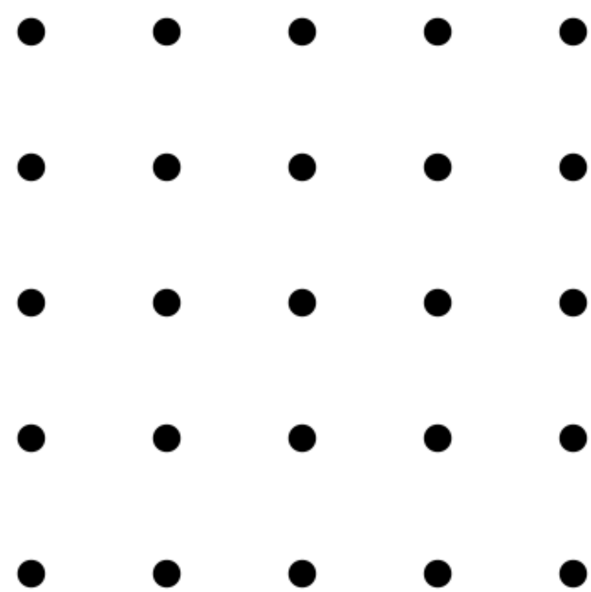
Puzzle of the Week

Finding Squares – 2

Drawn in red in this grid are two 1 by 1 squares and one 2 by 2 square with horizontal and vertical sides.



THE CHALLENGE: FInd the number of squares of all sizes and orientations in this grid. Unlike what is shown in the introduction, not all of these squares will have horizontal and vertical sides!



EXPLORATION: Play with what happens with larger grids and with grids that are not squares.

Puzzle of the Week
Finding Squares – 2 – Notes

THE CHALLENGE: The possible squares with horizontal and vertical sides have side lengths from 1 up to one less than the number of rows/columns.

For each size of square, think of the upper left corner as the starting position. You can produce all squares of this size in the grid by shifting this starter square around. You have the choice of moving it to the right or down the remaining number of positions. These are independent choices, so you get <this number plus 1> squared of that size square.

For example, suppose you are counting 2 by 2 squares in this 5 by 5 grid. A 2 by 2 square in the upper left-hand corner can be shifted 1 or 2 to the right and 1 or 2 down. So there are 3 possible positions horizontally (counting the original position) and 3 possible positions vertically. Thus there are 3 x 3 of these 1 by 1 squares.

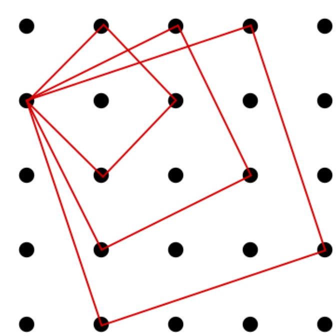
For the original 5 by 5 grid problem, this gives the following count of squares with horizontal and vertical sides:

- 1 by 1 squares: 4 x 4 = 16
- 2 by 2 squares: 3 x 3 = 9
- 3 by 3 squares: 2 x 2 = 4
- 4 by 4 squares: 1 x 1 = 1.

There are a total of 16 + 9 + 4 + 1 = 30 possible squares with horizontal and vertical sides.

The trick for the remaining squares is to continue to be organized in your counting. I'll describe a square by what one of its sides does. For example, a square with a side that goes one to the right and one up, or a square with a side that goes two the right and one up - these are two of the squares pictured in this illustration.

Counting by sliding as before we have:



- 1 to the right & 1 up: 3 x 3 = 9
- 2 to the right & 1 up: 2 x 2 = 4
- 3 to the right & 1 up: 1 x 1 = 1
- 2 to the left & 1 up: 2 x 2 = 4
- 3 to the left & 1 up: 1 x 1 = 1
- 2 to the right and 2 up: 1 x 1 = 1

So the total is: 30 (from earlier) + 9 + 4 + 1 + 4 + 1 = 49.

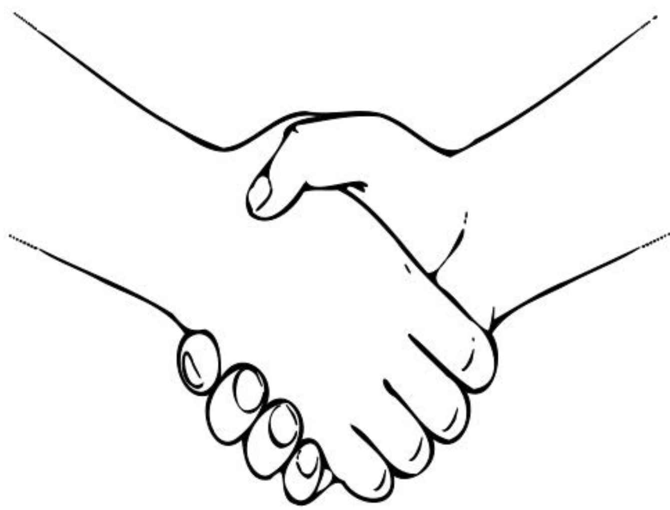
EXPLORATION: You can explore other sizes by finding an organized system of counting!

Puzzle of the Week

Handshakes at a Party – 1

Ten people were at a party. A lot of handshaking took place. When asked how many hands each person shook, the answers were: 3, 3, 4, 4, 4, 5, 5, 5, 6, and 8. When this was announced to the group, one person yelled out "That's impossible!"

THE CHALLENGE: Was that person right, did someone make a mistake in their handshake count? How do you know?



EXPLORATION: Look further into handshake counts and come up with a short list of counts that are possible and some that are impossible.

Puzzle of the Week

Handshakes at a Party – 1 – Notes

THE CHALLENGE: An interesting property of handshakes is that they are mutual - if I shake your hand, you are shaking my hand as well. This type of property can come up in other contexts, such as friendships that are mutual.

Because of this property of handshakes, if you total up the handshakes for all the people involved in an event, every handshake will be counted twice, once for each side of the handshake. Consequently, there should be an even number when all the handshakes are added up. If you add up 3, 3, 4, 4, 4, 5, 5, 5, 6, and 8, the total is 47, which is an odd number. So, a mistake must have been made!

EXPLORATION: From what was just discussed, if a collection of counts is going to be possible, their sum must be even. Another obvious requirement is that each individual count must be less than the total number of people (no one shakes their own hand).

However, those two conditions are not enough. For example, consider the counts 4, 4, 4, 4, 2. Of these five people, four of them have shaken everyone else's hand, so it is not possible for the last person to only have two handshakes. In a similar way, while 4, 4, 4, 3, 3 could happen, 4, 4, 4, 2, 2 cannot.

In "Handshakes at a Party - 2" we will see that the handshake list 7, 6, 5, 4, 3, 2, 1, 0 is impossible, despite having an even sum.

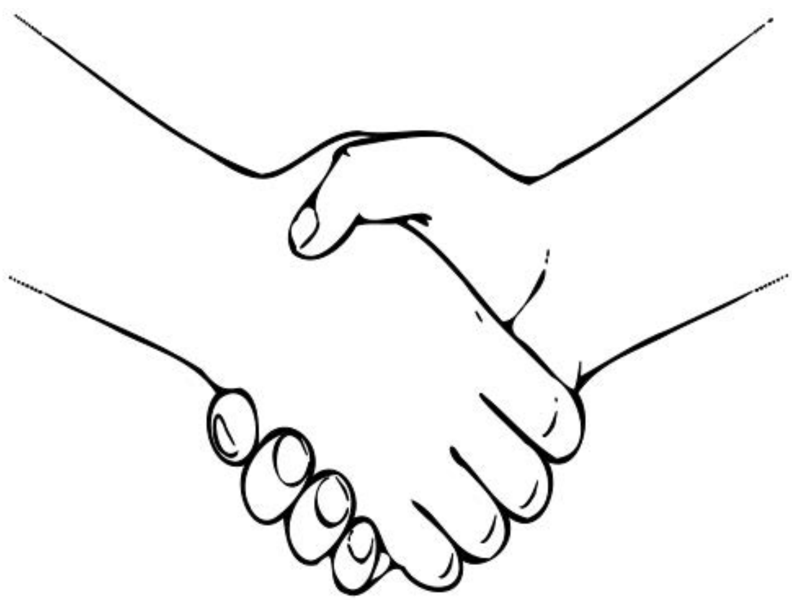
This leaves plenty of room to be explored further by the interested student.

Puzzle of the Week

Handshakes at a Party – 2

Eight people were at a party. A lot of handshaking took place. When asked how many hands each person shook, they were amazed to discover that each number was different. When this was announced to the group, one person yelled out “That’s impossible!”

THE CHALLENGE: Was that person right, did someone make a mistake in their handshake count? How do you know?



Puzzle of the Week

Handshakes at a Party – 2 – Notes

THE CHALLENGE: No one shakes their own hand. Therefore, for eight people, the maximum number of handshakes for any one person is seven. If the eight numbers are different, the list of numbers must be exactly the numbers from 0 to 7.

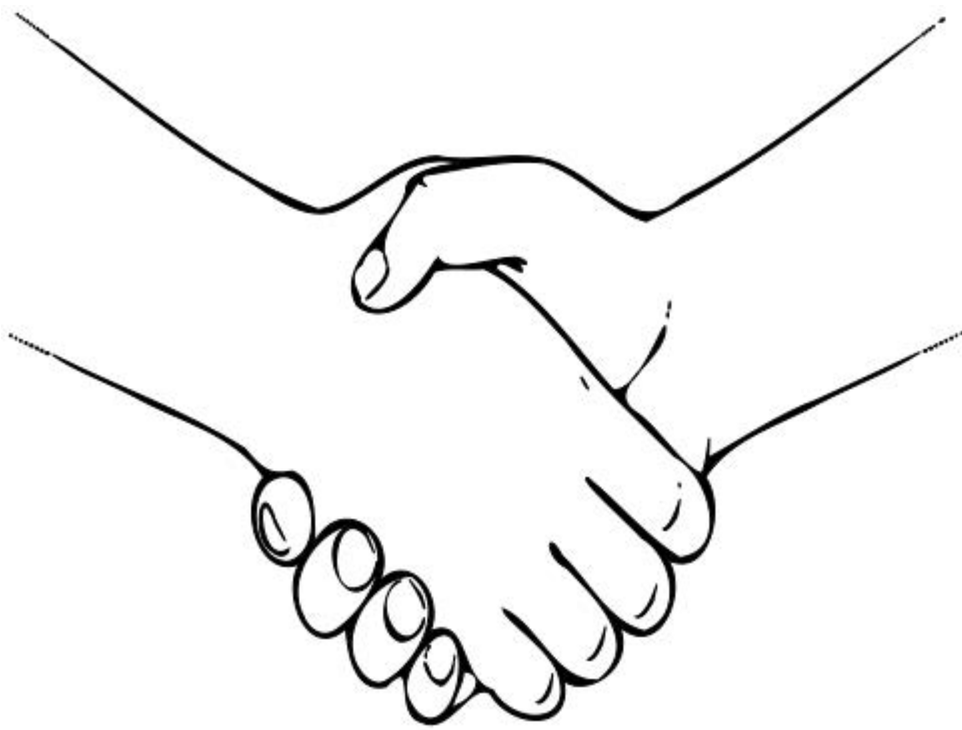
However, this cannot happen. If one person shook seven hands, then they shook every one else's hand at the party. That means each person was involved in at least one handshake, and that rules out the possibility of 0 handshakes. It is impossible to have all eight numbers from 0 to 7!

Puzzle of the Week

Handshakes at a Party – Couples

There were four married couples at a party. A lot of handshakes took place. No one shook the hand of their spouse. One person, Sam, was surprised when Sam asked the seven other people how many handshakes they made - the seven handshake counts were different!

THE CHALLENGE: How is this possible, and how many handshakes did Sam's spouse make?



Puzzle of the Week

Handshakes at a Party – Couples – Notes

THE CHALLENGE: No one shook their own hand, and if they didn't shake their spouse's hand, then the maximum count for any person was six. If the seven counts were different, then they must exactly be the whole list of numbers from 0 to 6.

Consider the person with six handshakes. They shook everyone's hand except their spouse. Put another way, we know everyone other than that person's spouse had at least one handshake. Therefore, the person with 0 handshakes had to be married to the person with six handshakes!

If you remove this couple and the handshakes they were involved in, you are left with three married couples, and each of these six people will have their handshake count reduced by one. We now have a new problem very similar to the original one. These three couples will have handshake counts that are exactly the list of numbers from 0 to 4 (not listing Sam's count). For the same reasons as before, we can be sure that the person with four handshakes on this list (five handshakes originally) is married to the person with zero handshakes on this list (one handshake originally).

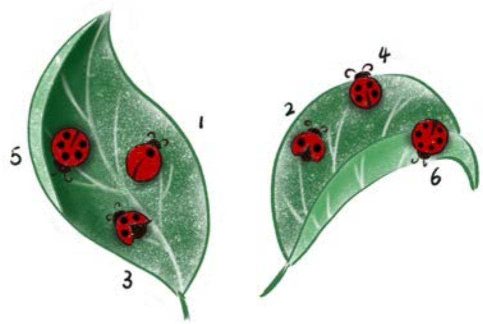
Carrying out this logic one more time, we come to the conclusion that 6 and 0 are married, 5 and 1 are married, and 4 and 2 are married. The only remaining person had 3 handshakes, and they must be married to Sam!

By the way, Sam had three handshakes as well (Sam shook hands with 4, 5, and 6).

Puzzle of the Week

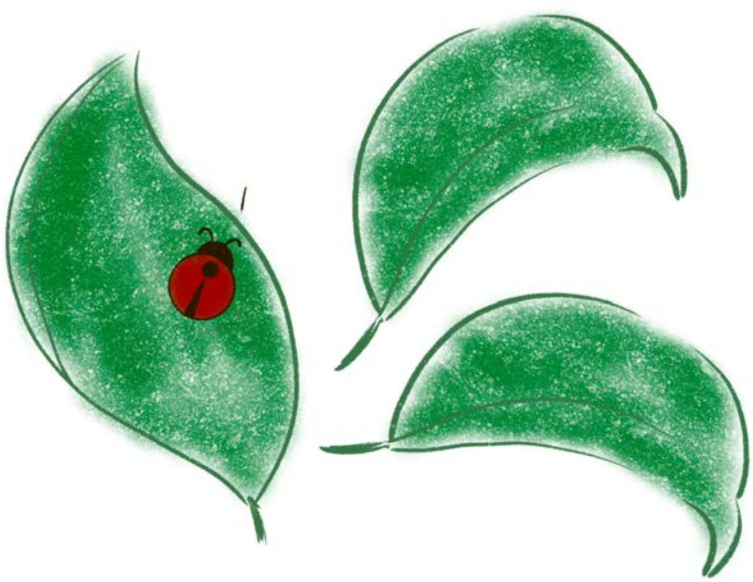
Ladybugs that don't Add Up – 2

Numbered ladybugs are landing on two leaves. The rule is: the number of dots of two ladybugs on a leaf cannot add up to the number of dots on another ladybug on that leaf. The leaf on the left is fine, but the leaf on the right has 2 + 4 = 6.



THE CHALLENGE: Starting at 1 and counting up, how high can you go putting the numbered ladybugs on three leaves while following the rule for each of the leaves.

EXPLORATION: How do things change if you use only even numbers? How do things change if you use only odd numbers? What happens if you use more than three leaves?



Puzzle of the Week

Ladybugs that don't Add Up – 2 – Notes

THE CHALLENGE & EXPLORATION: Three leaves is a lot trickier to deal with in an organized way than two leaves.

While there is much that is complex about this problem, there are two simple ways to put numbers in a leaf that are surprisingly powerful: 1) put the powers of 2 on the leaf (1 2 4 8 16 …) and 2) starting with some number, put all the consecutive numbers up through twice that number (5 6 7 8 9 10).

We can also take the best answer for two leaves, (1 2 4 8) - (3 5 6 7), and then put the next numbers on the third leaf: (1 2 4 8) - (3 5 6 7) - (9 10 11 12 13 14 15 16 17 18). After that good start, we can put a few more numbers on the first two leaves: (1 2 4 8 22) - (3 5 6 7 19 20 21) - (9 10 11 12 13 14 15 16 17 18).

That answer, that goes up to 22, is surprisingly close to the best answer. To know for sure what the best possible answer is, you either need to write a computer program or you can consult the math literature on "sum-free sets" or "sum-free partitions." In the literature you will find the following answer to 23, which has only a couple changes (moving the 11 and 16)from the earlier answer: (1 2 4 8 11 16 22) (3 5 6 7 19 21 23) (9 10 12 13 14 15 17 18 20).

The best answer for four leaves goes up to 66 and is: (1 2 4 8 11 16 22 25 40 43 53 66) (3 5 6 7 19 21 23 34 35 50 51 52 63 64 65) (9 10 12 13 14 15 17 18 20 54 55 56 57 58 59 60 61 62) (24 26 27 28 29 30 31 32 33 36 37 38 39 41 42 44 45 46 47 48 49). You can get surprisingly close to this answer by employing the same strategy as we used for getting a first answer for three leaves.

The best answer for five leaves goes up to 196. It is an unsolved problem what the best answer is for six leaves! Perhaps one of your students will have fun playing with this and coming up with a great answer!

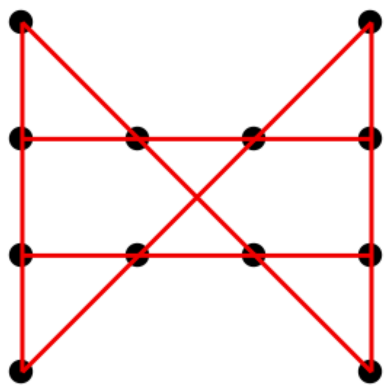
Working with even numbers is simply a matter of doubling everything from before. The best answer for even numbers will be twice the best answer for regular numbers.

The odd numbers can all be put on one leaf, so having three leaves doesn't make much difference for them.

Puzzle of the Week

Lines – 3

Here is an example of 12 dots with 6 straight line segments so that each line segment goes through exactly 4 dots.

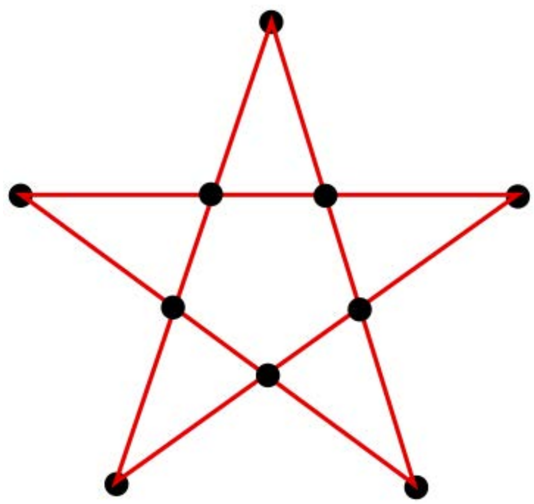


THE CHALLENGE: Design an arrangement of 10 dots with 5 straight line segments so that each line segment goes through exactly 4 dots.

Puzzle of the Week

Lines – 3 – Notes

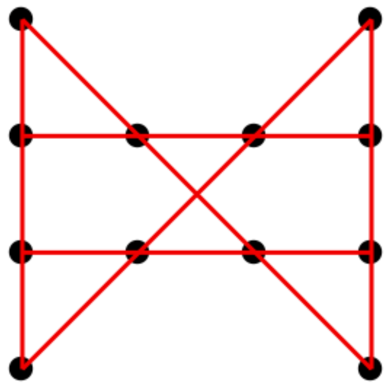
THE CHALLENGE: A star shape works well for this. Here is an answer. Perhaps one of your students will come up with a different answer?



Puzzle of the Week

Lines – 4

Here is an example of 12 dots with 6 straight line segments so that each line segment goes through exactly 4 dots.

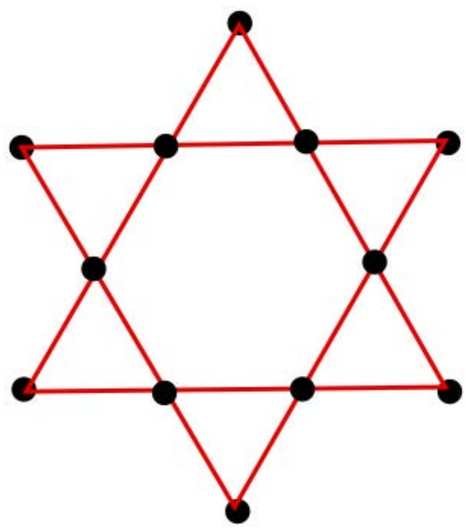


THE CHALLENGE: Design a completely different arrangement of 12 dots with 6 straight line segments so that each line segment goes through exactly 4 dots.

Puzzle of the Week

Lines – 4 – Notes

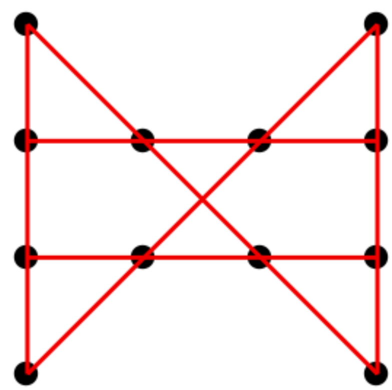
THE CHALLENGE: A star shape works well for this. Here is an answer. Perhaps one of your students will come up with a different answer?



Puzzle of the Week

Lines – 5

Here is an example of 12 dots with 6 straight line segments so that each line segment goes through exactly 4 dots.

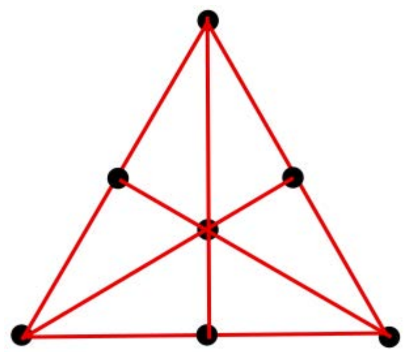


THE CHALLENGE: Design an arrangement of 7 dots with 6 straight line segments so that each line segment goes through exactly 3 dots.

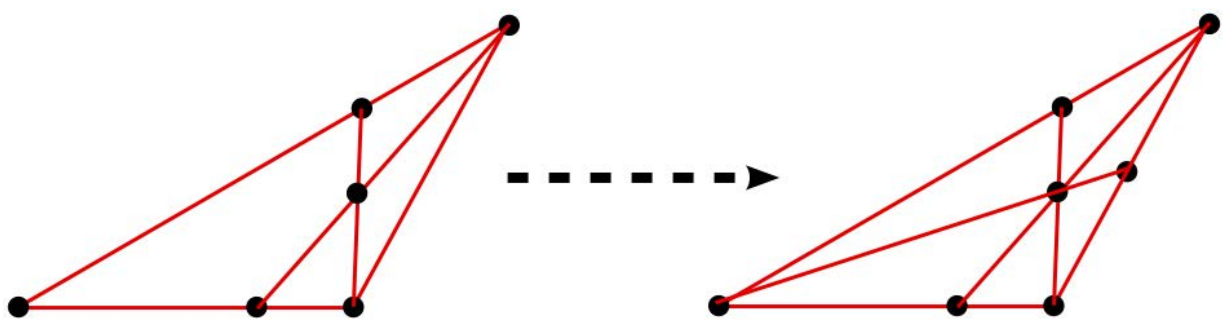
Puzzle of the Week

Lines – 5 – Notes

THE CHALLENGE: The answer involves putting a point inside a triangle. Once you think of this, it is tempting to create an equilateral triangle and have the three internal lines meet as in this diagram.



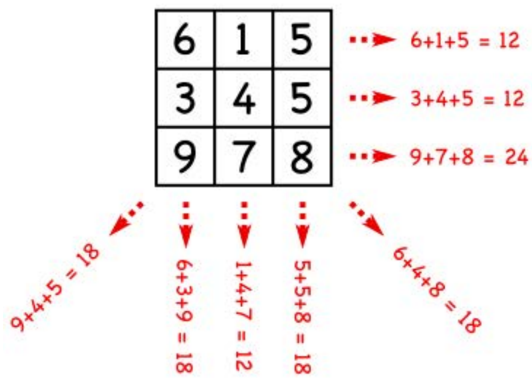
However, lest you think the equilateral triangle is important, there is no need for that much care. Take any triangle, put two internal lines across from the corners, and then create the last line to go through the remaining corner and the intersection point inside the triangle.

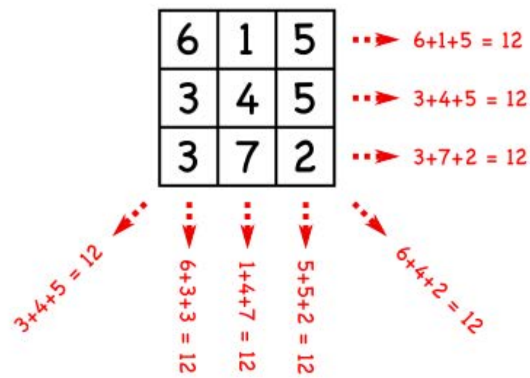


Puzzle of the Week

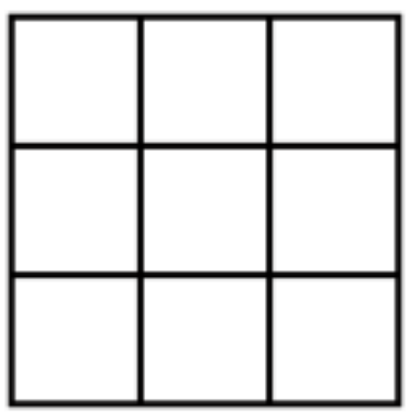
Magic Squares – 3

In a *Magic Square*, all the rows, columns and diagonals add up to the same number. This first square is <u>not</u> a Magic Square. The second one is a Magic Square with a constant sum of 12.





THE CHALLENGE: Use each of the numbers from 0 to 8 once to complete this Magic Square.



0 1 2 3 4 5 6 7 8

EXPLORATION: Can you find more than one way to do it? What do the different ways have in common? How would your answer change if you used the numbers from 1 to 9? How about the even numbers from 2 to 18?

Puzzle of the Week

Magic Squares – 3 – Notes

THE CHALLENGE: Let your students play with this. If they pay attention to what they're doing, they'll discover interesting relationships and get a lot out of it. For young students, there is absolutely no need to go into any kind of careful analysis. What follows is a more analytical way to find the solutions.

Common Sum: The simplest way to start analyzing this puzzle is to find the common sum. Each row adds up to the common sum. Also, the three rows contain the numbers from 0 to 8 and add up to three times the common sum. Therefore, three times the common sum is 36 (the sum of 0 to), so the common sum is 12.

Central Square: The next step is to add up the four lines that go through the center square. The common sum is 12 and there are four lines, so their sum must be 4 x 12 = 48. Alternatively, the four lines contain every number once, plus the central square three more times. The sum of the numbers from 0 to 8 is 36. So, 48 equals 36 plus 3 times the central square. So, the central square must be 4.

Adding up to 12: There are surprisingly few ways to add up to 12. They are:

(0 4 8) (1 4 7) (2 4 6) (3 4 5) (0 5 7) (1 3 8) (1 5 6) (2 3 7)

You can figure out a lot for this list. Look at how often a number appears in a triplet:

- 4 times: 4
- 3 times: 1, 3, 5, 7
- 2 times: 0, 2, 6, 8

Next, compare this to how many times a square in the diagram is in one of the lines. You'll see that the center square is in four lines, the corner squares are in three lines, and the middle of the sides are in two lines. This is another way to see that the center square must be 4. Also, the corners must be 1, 3, 5, and 7, and the middle of the sides must be 0, 2, 6, and 8.

Fill up the Square: The hard work is done. Start with 4 in the middle and put 7 in one corner. Note that 0 must go next to the 7 on one side or the other (otherwise the 8 would be forced next to the 7) . You will have no choices after that. One answer, by rows, is: (7 0 5) (2 4 6) (3 8 1). Notice that this is the same as any other answer by rotating the square and possibly flipping it.

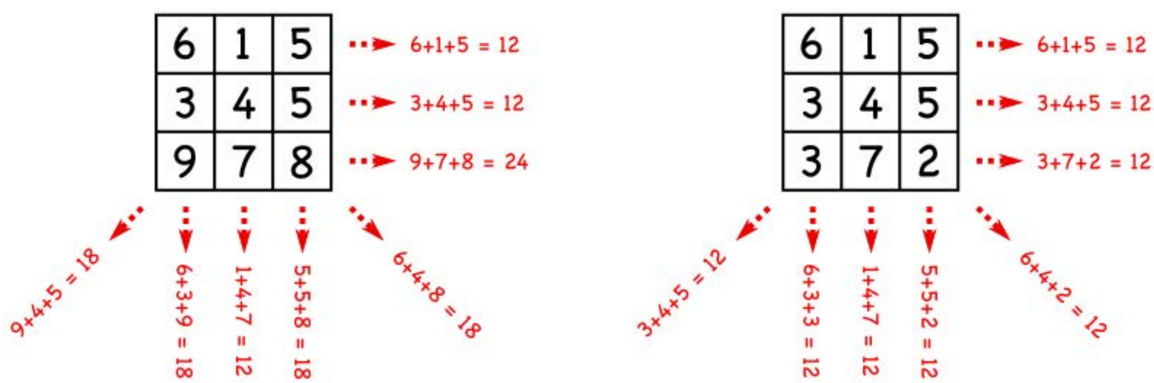
EXPLORATION: As noted in the last paragraph, all the solutions are essentially the same - rotate the square until the 7's are in the same corner, and take the mirror image (if needed) along the diagonal to put the 0 in the same position.

Solving this puzzle for 1 to 9 would mean adding 1 to every entry in the 0 to 8 solution. Solving this puzzle for 2 to 18 would mean doubling all the entries for the 1 to 9 solution.

Puzzle of the Week

Magic Squares – 4

In a ***Magic Square***, all the rows, columns and diagonals add up to the same number. This first square is <u>not</u> a Magic Square. The second one is a Magic Square with a constant sum of 12.



THE CHALLENGE: Fill in these two Magic Squares using these two sets of numbers: 1) {6, 7, 8, 9, 10, 11, 12, 13, 14} and {3, 6, 9, 12, 15, 18, 21, 24, 27}.

EXPLORATION: Look at all the Magic Squares you've seen that don't have duplicate entries. Make a table of your results and look for patterns. If you were given any other sequence of numbers that are evenly spaced, such as {4, 9, 14, 19, 24, 29, 34, 39, 44}, would you be able to immediately fill in the Magic Square?

Puzzle of the Week

Magic Squares – 4 – Notes

THE CHALLENGE & EXPLORATION: Start with the solution to the standard 1 - 9 version of this puzzle.

8	1	6
3	5	7
4	9	2

To make a Magic Square with 6 to 14, simply add 5 to all of those entries. To make a Magic Square for {3, 6, 9, 12, 15, 18, 21, 24, 27}, simply triple all the entries of the original solution.

13	6	11
8	10	12
9	14	7

24	3	18
9	15	21
12	27	6

Let's see why that works. Suppose we have a + b + c = S, where S is the sum used for all rows, columns, and diagonals. In the original 1 - 9 puzzle, S was 15.

If we add 5 to all the entries, (a + 5) + (b + 5) + (c + 5) = (a + b + c) + 15 = S + 15. The entries in the new puzzle all add up to a sum that is 15 more than the original, so they still all add up to the same (new) thing.

If we multiply by 3, then (3a) + (3b) + (3c) = 3(a + b + c) = 3S. The entries in the new puzzle add up to a sum that is three times the original.

If we do any combination of multiplying and adding, it still works. Suppose we multiply by m and then add n. We get (ma + n) + (mb + n) + (mc + n) = (ma + mb + mc) + (n + n + n) = m (a + b + c) + 3n = mS + 3n. So, no matter what m and n are, the new entries end up all adding up to the same thing!

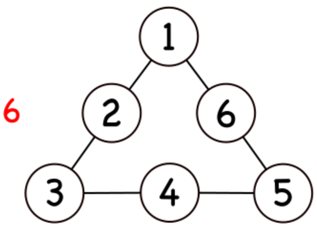
Of course, your students are not ready for this algebra, but by going through these examples they will see the patterns of how this works.

Puzzle of the Week

Magic Triangles – 1

The sums of the sides of a *Magic Triangle* are all the same. This example is **NOT** a Magic Triangle.

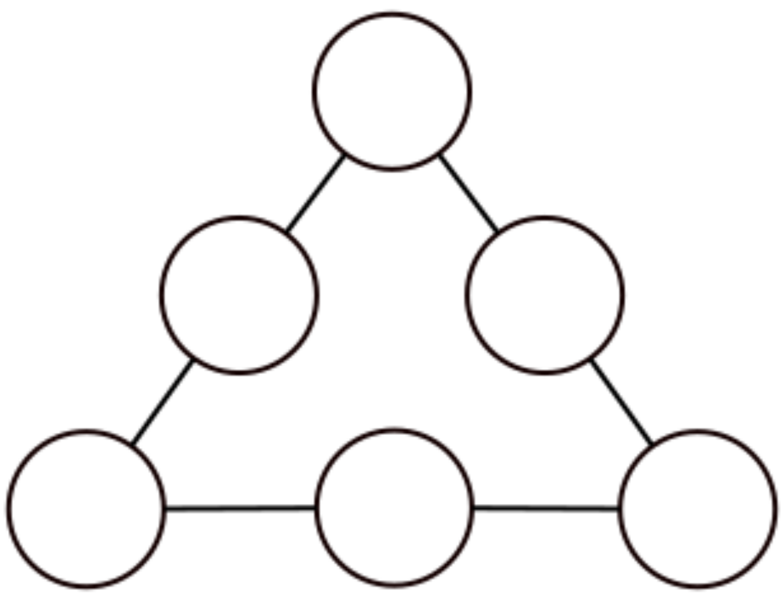
1 + 2 + 3 = 6



1 + 6 + 5 = 12

3 + 4 + 5 = 12

THE CHALLENGE: Use the numbers from 1 to 6 to make a Magic Triangle below.



1 2 3 4 5 6

EXPLORATION: What are the different sums that are possible for Magic Triangles that use 1 to 6?

Puzzle of the Week

Magic Triangles – 1 – Notes

THE CHALLENGE & EXPLORATION: Let your students play with this. If they pay attention to what they're doing, they'll discover interesting relationships and get a lot out of it. For young students, there is absolutely no need to go into any kind of careful analysis.

To be more analytical, add up the three sides. This sum will be the sum of the numbers from 1 to 6 plus the three corners an extra time. The sum of the numbers from 1 to 6 is 21. So, three times the common sum is 21 plus the sum of the three corners. Looked at another way, the common sum will be 7 plus one third of the sum of the corners. The smallest possible sum of three corner numbers is 1 + 2 + 3 = 6, and the largest is 4 + 5 + 6 = 15. So, the common sum might be anything from 7 + (6 / 3) = 9 to 7 + (15 / 3) = 12. Let's look at them 1 at a time.

Common Sum = 9. The corners must be 1, 2, and 3. The number between 1 and 2 must be 6. The number between 1 and 3 must be 5. The number between 2 and 3 must be 4. It works!

Common Sum = 10. The corners add up to 9. The corners could be (1 3 6), (1 3 5), or (2 3 4). (1 2 6) cannot work because there is nothing that can be put between 1 and 2. (1 3 5) can work by putting 6 between 1 and 3, 4 between 1 and 5, and 2 between 3 and 5. (2 3 4) cannot work because there is nothing that can be put between 2 and 4.

Common Sum = 11. The corners add up to 12. The corners could be (1 5 6), (2 4 6), or (3 4 5). (1 5 6) cannot work because there is nothing to put between 5 and 6. (2 4 6) can work because you can put 1 between 4 and 6, 3 between 2 and 6, and 5 between 2 and 4. (3 4 5) cannot work because there is nothing to put between the 3 and 4.

Common Sum = 12. The corners must be 4, 5, and 6. The number between 4 and 5 must be 3. The number between 4 and 6 must be 2. The number between 5 and 6 must be 1. It works!

There are four solutions in total.

You can save a lot of work if you realize that you can get all the answers for the Common Sum being 11 and 12 by taking the answers for the Common Sum being 9 and 10 and subtracting those entries from 7. For example, the answer for Common Sum = 9 has sides (1 6 2), (1 5 3), and (2 4 3). If these entries are subtracted from 7, the answer for Common Sum = 12 is found, namely (6 1 5), (6 2 4), and (5 3 4).

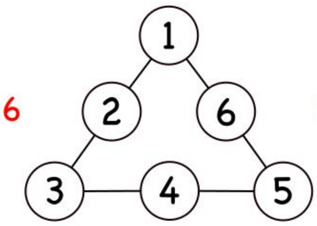
If you compare "Equal Sums – 2" with "Magic Triangles – 1," you will see that they are the same puzzle!

Puzzle of the Week

Magic Triangles – 2

The sums of the sides of a *Magic Triangle* are all the same. This example is **NOT** a Magic Triangle.

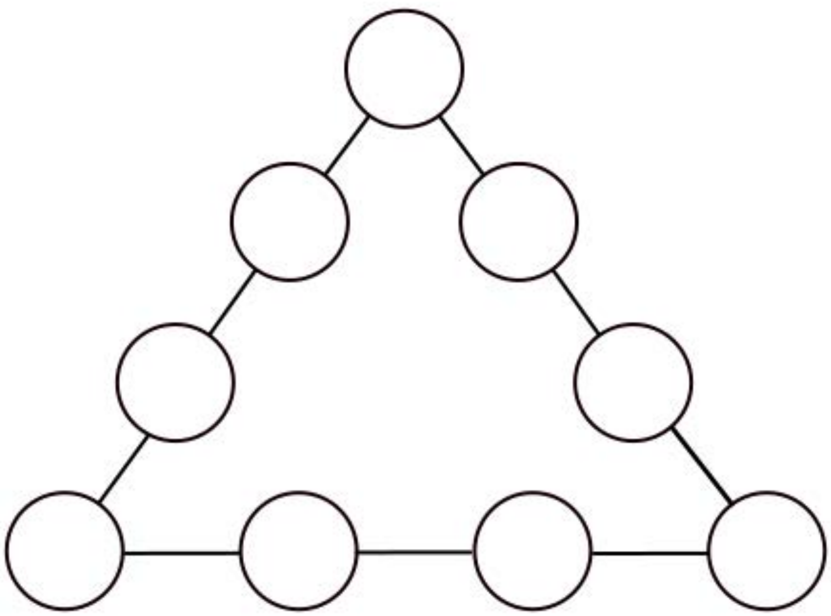
1 + 2 + 3 = 6



1 + 6 + 5 = 12

3 + 4 + 5 = 12

THE CHALLENGE: Use the numbers from 1 to 9 to make Magic Triangles.



1 2 3 4 5 6 7 8 9

EXPLORATION: What are the different sums that are possible for Magic Triangles that use 1 to 9?

Puzzle of the Week

Magic Triangles – 2 – Notes

THE CHALLENGE & EXPLORATION: Let your students play with this. If they pay attention to what they're doing, they'll discover interesting relationships and get a lot out of it. For young students, there is absolutely no need to go into any kind of careful analysis.

To be more analytical, add up the three sides. This sum will be the sum of the numbers from 1 to 9 plus the three corners an extra time. The sum of the numbers from 1 to 9 is 45. So, three times the common sum is 45 plus the sum of the three corners. Looked at another way, the common sum will be 15 plus one third of the sum of the corners. The smallest possible sum of three corner numbers is 1 + 2 + 3 = 6, and the largest is 7 + 8 + 9 = 24. So, the common sum might be anything from 15 + (6 / 3) = 17 to 15 + (24 / 3) = 23. The Common Sum can be 17 to 23.

As noted at the end of the Notes on Magic Triangles – 1, we can cut our work in half by using the answers from 17, 18, 19 and 20 to give us answers for 20, 21, 22, and 23 by subtracting all the entries from 10.

Common Sum = 17. The corners are (1 2 3). The sides of one solution are (1 5 9 2), (1 6 7 3), and (2 4 8 3). Another solution is (1 6 8 2), (1 4 9 3), and (2 5 7 3).

Common Sum = 18. The corners add up to 9. The corners can be (1 2 6), (1 3 5), and (2 3 4), but none of them work out.

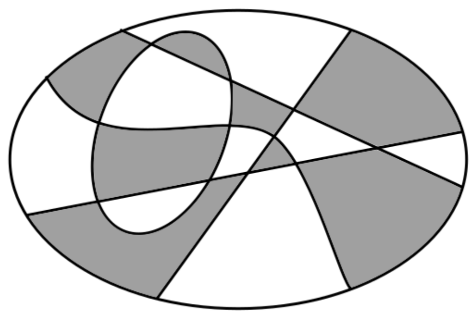
Common Sum = 19. The corners add up to 12. The corners can be (1 2 9), (1 3 8), (1 4 7), (1 5 6), (2 3 7), (2 4 6), and (3 4 5). (1 2 9) and (1 3 8) do not work. For (1 4 7) we have the solution (1 6 8 4), (1 2 9 7), and (4 3 5 7). Needless to say, there is a lot to look at if you want to look through every possibility.

Common Sum = 20. The corners add up to 15. There are even more possibilities here, with no obvious way to shorten the search list.

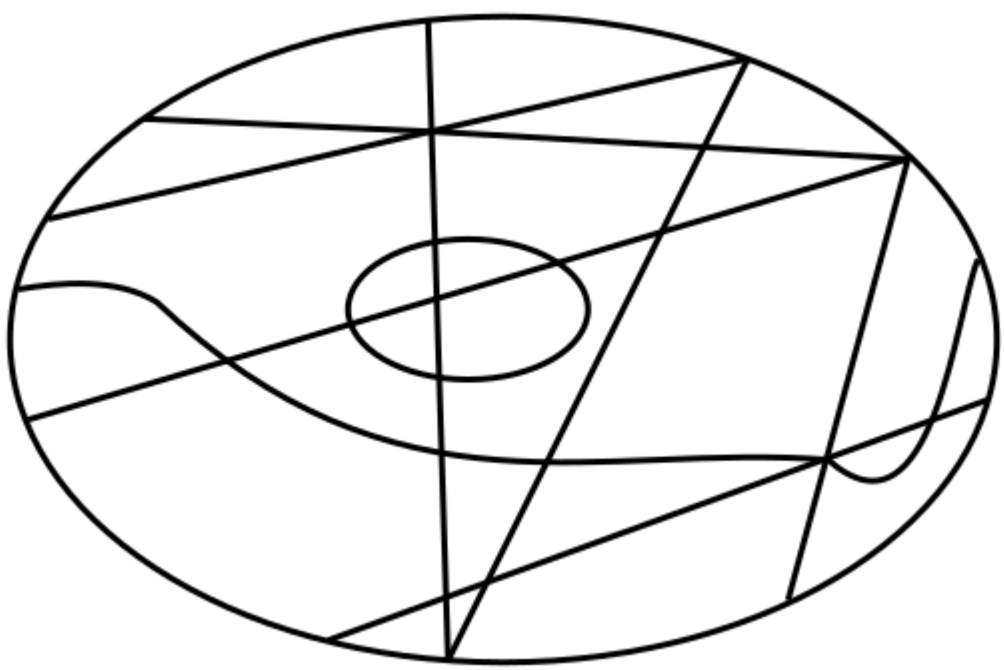
Puzzle of the Week

Map Coloring with 2 Colors – 2

Map makers color maps so regions sharing a border have different colors. Maps that have an even number of lines going out from each interior intersection point can always be colored using 2 colors. Here is an example.



THE CHALLENGE: This more complicated map has an even number of lines going out from each interior intersection point. Color it using just two colors.



EXPLORATION: Create a simple map with an interior intersection point that has an odd number of lines going out from it. Why is it impossible to color your map with just 2 colors?

Puzzle of the Week

Map Coloring with 2 Colors – 2 – Notes

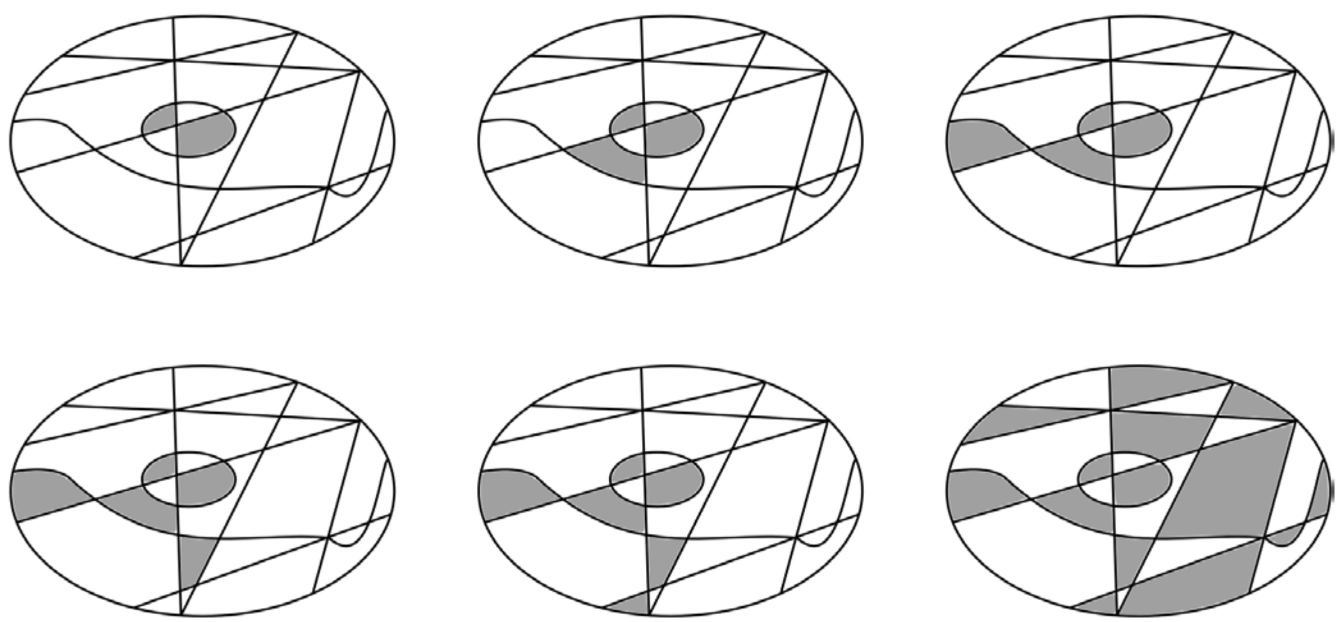
THE CHALLENGE: This is easier than it sounds. The only trick is to not jump around in the map.

Start at any intersection point. For all the regions that have that point as a corner, color the regions with alternating colors as you work your way around the point in one direction.

Once you have finished with a point, move to any other point that has at least one region already colored. Starting with the region that was already colored, work your way around this new point alternating colors as you go.

Because there is an even number of regions that have a common corner, you will always be able to alternate the colors when working your way around an intersection point.

It's as simple as that!

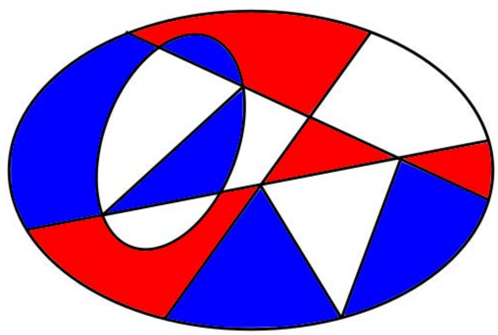


EXPLORATION: If you have an odd number of regions surrounding a point and you try to use alternating colors, you will get stuck as you work your way around the point. At the end, you will have to use a new color for the last region!

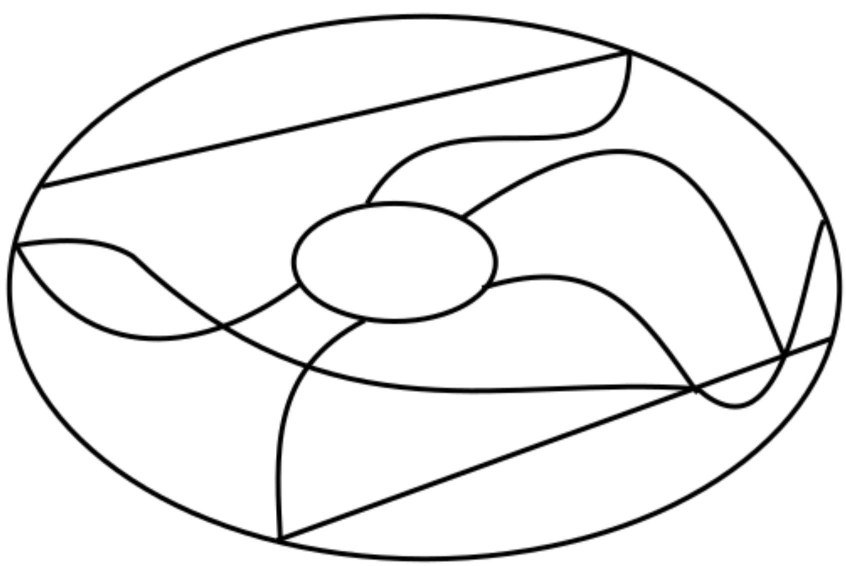
Puzzle of the Week

Map Coloring with 4 Colors

Map makers color maps so regions sharing a border have different colors. Mathematicians have shown that every map that has connected regions can be colored with four or fewer colors. In a previous puzzle, we saw that if there is a place on a map that has an odd number of lines coming out from it, then the map will need at least three colors. Here is an example of such a map.



THE CHALLENGE: Color this map using as few colors as you can.

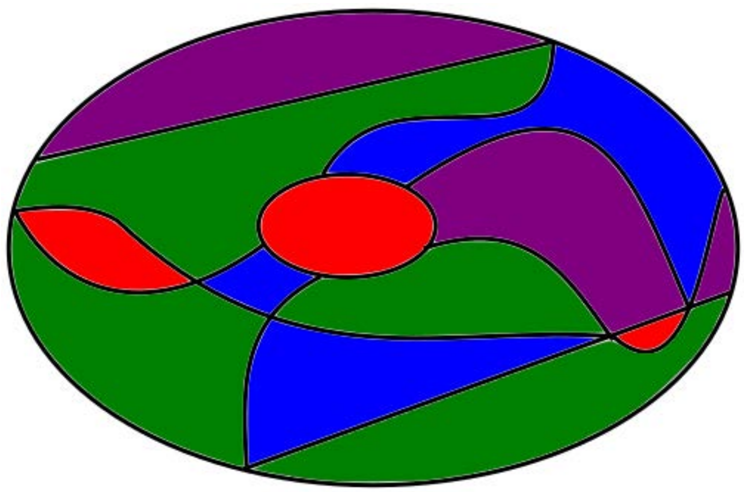


EXPLORATION: Create the simplest map you can that requires four colors. What happens if you have a map with some “regions” made up of completely separate pieces. These multi-piece regions still must be a single color. Make a map with some multi-piece regions that needs more than four colors to color.

Puzzle of the Week
Map Coloring with 4 Colors – Notes

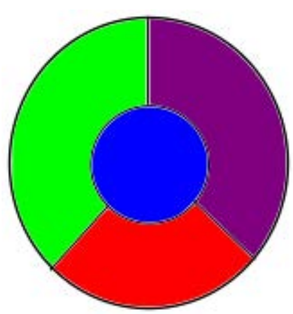
THE CHALLENGE: In “Map Coloring with 2 Colors - 2” we saw that a single corner that has an odd number of regions around it will require three colors to color those regions.

Well, if we have a single region with an odd number of regions around it, that will force the use of four colors. If you look at the oval at the center of the second map on the previous page, that oval has an odd number of regions around it and that will force the use of four distinct colors to color everything.



This happens on the map of the United States. The state of Nevada has five states surrounding it, so those six states (including Nevada) will always require at least four colors.

EXPLORATION: The following map forces the use of four colors and is as simple as you can get.



If you allow disconnected regions, you can force as many colors as there are regions - you just construct it so that every region touches every other region. Because every region can be disconnected, there aren’t any true physical limitations.

Puzzle of the Week

Moving Digits – 1

Any multiple of 10 has the property that if we remove the ones digit, the new number evenly divides the original number. Take 50 for example. If you remove the 0 you get 5, and 5 evenly divides 50.

THE CHALLENGE: Of the two-digit numbers that are not multiples of 10, which ones have the property that if their ones digit is deleted, the new number evenly divides the original number?

When does A divide AB?

EXPLORATION: Other than multiples of 10, investigate why there are no numbers larger than 100 that have this property.

Puzzle of the Week

Moving Digits – 1 – Notes

THE CHALLENGE: It is straightforward to simply list all the numbers: 11, 12, 13, 14, 15, 16, 17, 18, 19, 22, 24, 26, 28, 33, 36, 39, 44, 48, 55, 66, 77, 88, and 99.

EXPLORATION: What follows is some formal algebra to justify why it is impossible. Your students may come up with reasoning that involves numbers and examples that use reasoning equivalent to this algebra, and that would be wonderful!

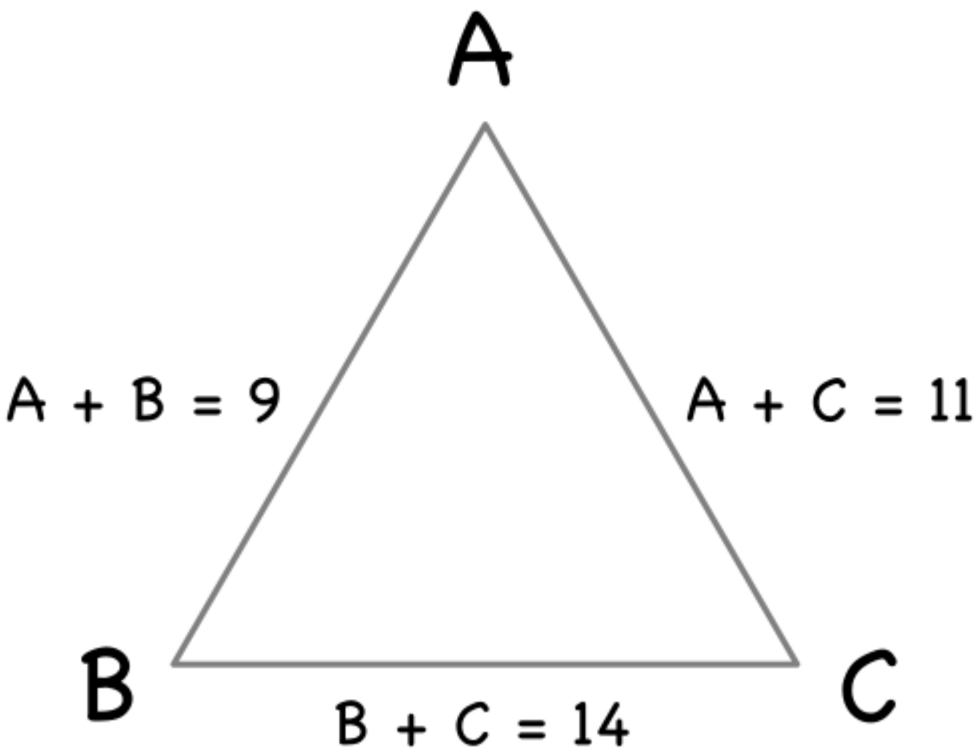
For example, a student might look at a number such as 127. If you divide 127 by 12, you get 10 + 7/12. After doing several such examples, they can reasonably claim that it can never work for numbers larger than 100.

Algebra: Write the number as 10a + b, where b is the ones digit (we are assuming b is not 0). We want to find numbers where a evenly divides 10a + b. We are assuming that a evenly divides 10a + b. So, if we divide 10a + b by a, we get 10 + b/a, and a must evenly divide b. Because b is between 1 and 9, that forces a to be less than 10 and we are done.

Puzzle of the Week

Mystery Sums – 1

THE CHALLENGE: There are secret numbers at the corners of a triangle. The sums of these numbers are written in the middle of each side. The numbers in the middle of the sides are: 9, 11, and 14. Find the secret numbers.



EXPLORATION: Make up a few of these problems for your friends. Can you come up with a general method for solving this kind of puzzle? Can you solve these for polygons with more sides?

Puzzle of the Week

Mystery Sums – 1 – Notes

THE CHALLENGE & EXPLORATION: By adding up all three of the middles, you will find twice the sum of all three numbers. If you then divide that by two, you will have the sum of all three numbers. If you then subtract each of the middle numbers from this sum, you will find the unknown numbers.

In our problem, (9 + 11 + 14) / 2 = 34 / 2 = 17. So the corner opposite the side with the 9 is 17 - 9 = 8. The corner opposite the 11 is 17 - 11 = 6, and the side opposite the 14 is 17 - 14 = 3.

For four-sided figures, there is not enough information to recover the original numbers on the corners. For example, the four corners {9, 4, 3, 7} produce the same sum information in the middle of the sides as {8, 5, 2, 8} and {7, 6, 1, 9}. For more than four sides, the situation gets even worse. Not every investigation produces exciting results, but it’s good to explore.

We’ll see in “Mystery Sums - 2” that the proper generalization of this triangle problem is to give all the sums for every possible pair of mystery numbers.

Puzzle of the Week

Mystery Sums – 2

THE CHALLENGE: There are five boxes to weigh and mail, and each of them weighs under 20 pounds. Unfortunately, the one available scale only weighs things over 20 pounds. The packages, weighed in pairs, weigh 22, 24, 25, 26, 27, 28, 29, 30, 32, and 33 pounds. How much does each package weigh?

22 24 25 26 27 28 29 30 32 33

EXPLORATION: Make up a few of these problems for your friends. See if you can come up with a general method for solving this kind of puzzle.

Puzzle of the Week

Mystery Sums – 2 – Notes

THE CHALLENGE & EXPLORATION: Each box is weighed four times in the process of weighing all the pairs of boxes. Therefore, if we add up all ten weights, we will find out what four times the total weight of the boxes is. If we divide this number by four, we will have the sum of their weights.

Adding up the weights we get 22 + 24 + 25 + 26 + 27 + 28 + 29 + 30 + 32 + 33 = 276. So, the total weight of the boxes is 276 / 4 = 69 pounds.

Let the weights of the boxes in order, from lightest to heaviest, be A, B, C, D, and E. Then A + B + C + D + E = 69. To get the smallest weight of a pair, we must add A and B, so A + B = 22. Similarly for the heaviest, D + E = 33. Consequently, 69 = (A + B) + C + (D + E) = 22 + C + 33 = 55 + C, which forces C = 14.

To get the second lightest weight of 24, we must add A and C, so 24 = A + C = A + 14, which forces A = 10. Similarly for the second heaviest weight of 32, we must add E and C, so 32 = E + C = E + 14, which forces E = 18.

To finish this off, A + B = 22 forces B = 12, and D + E = 33 forces D = 15.

So the weights in order are: 10, 12, 14, 15, and 18.

These steps will work for any set of five weights.

Puzzle of the Week
Odd Squares

An *Odd Square* is a square grid of numbers in which the numbers in each row and column add up to an odd number. This is a 3 by 3 Odd Square using the numbers from 3 to 11.

9	3	7	9+3+7= 19
6	4	5	6+4+5 = 15
8	10	11	8+10+11 = 29
9+6+8 = 23	3+4+10 = 17	7+5+11 = 23	

THE CHALLENGE: Use the numbers from 1 to 9 to form a 3 by 3 Odd Square.

1 2 3 4 5 6 7 8 9

EXPLORATION: Define what an Even Square would be and consider the Even Squares that can be formed by the numbers from 1 to 9.

Puzzle of the Week

Odd Squares – Notes

THE CHALLENGE: Note that a natural follow on puzzle after this is to do Prime Squares.

For a row and column to add up to an odd number, it must have 1 or 3 odd numbers in it. Because there are a total of 5 odd numbers, that means one row will have 3 odd numbers and two rows will have one odd number. Similarly, one column will have 3 odd numbers and two columns will have one odd number.

We can slide the rows and columns around and not really change the answer in an interesting way, so we might as well assume that the answer looks like this:

ODD	ODD	ODD
EVEN	EVEN	ODD
EVEN	EVEN	ODD

We can put in any odd numbers for the ODD places, and any even numbers for the EVEN places. Here is one possible answer for this puzzle, but any other distribution of odd and even numbers into the ODD and EVEN slots will work.

1	3	5
2	4	7
6	8	9

EXPLORATION: An Even Square would be a 3 by 3 square all of whose rows and columns added up to even numbers.

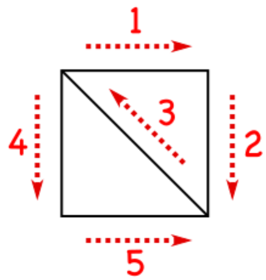
For a given row or column to add up to an even number, it would have to have 0 or 2 odd numbers in it.

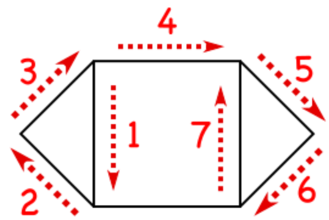
Looking at the rows, that means each row would have 0 or 2 odd numbers, and that would mean that the three rows combined would have 0, 2, 4, or 6 odd numbers. However, there are 5 odd numbers from 1 to 9, so making an Even Square with the numbers from 1 to 9 is impossible!

Puzzle of the Week

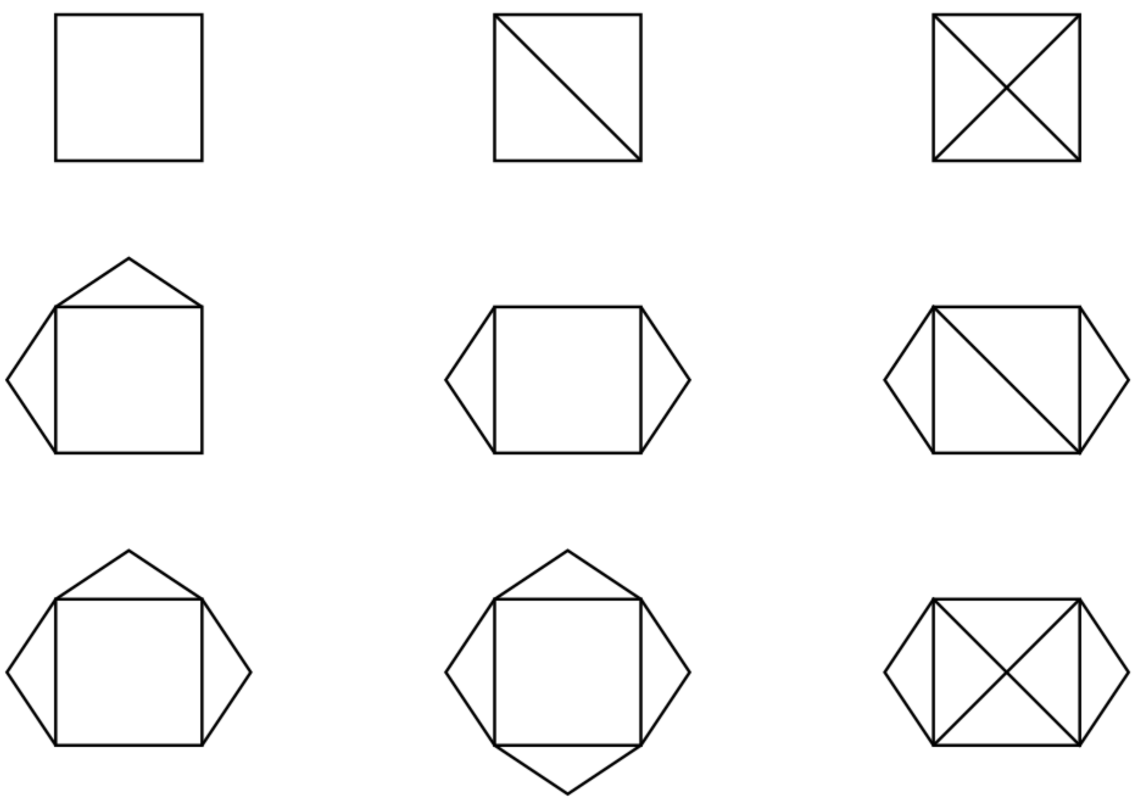
Parades – 1

Parades wish to visit each street on their route exactly once. In these two examples, the first one is a successful parade route and the second one is not (one street is left out).





THE CHALLENGE: For each street layout, either find a parade route that visits each street exactly once or decide that it is impossible. For street layouts that have a parade, which ones allow parades to start and end at the same place? Can you find a pattern in your results?



Puzzle of the Week

Parades – 1 – Notes

THE CHALLENGE: The key idea is to keep track of where parades start and end, if they exist at all for a street map. Some parades can start and end anywhere, and some must start or end at very specific locations.

After looking at lots of examples, the following observation emerges. It is not important for young children to prove these things.

Result 1: If a corner has an odd number of streets coming to it, the parade must start or end there.

The reason for this is simple. Every time a parade goes into and back out of a corner, that accounts for an even number of streets coming to that corner. Therefore, corners with an odd number of streets must be the start or end of the parade.

Result 2: If there are more than two corners with an odd number of streets coming into them, then this map cannot have a parade.

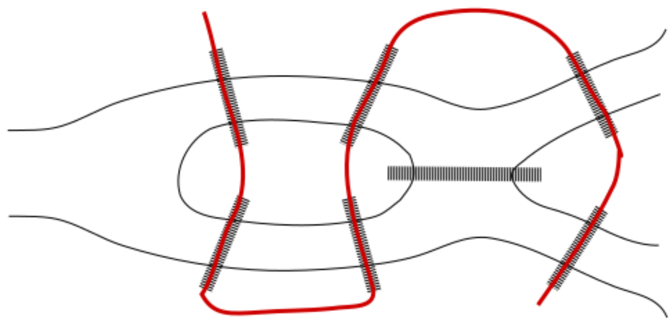
Result 3: If there are exactly two corners with an odd number of streets, then any parade must begin at one of them and end at the other. In particular, it is impossible to have a parade that begins and ends at the same place.

Result 4: If there are no corners with an odd number of streets, then a parade can start anywhere, and it must start and end at the same place.

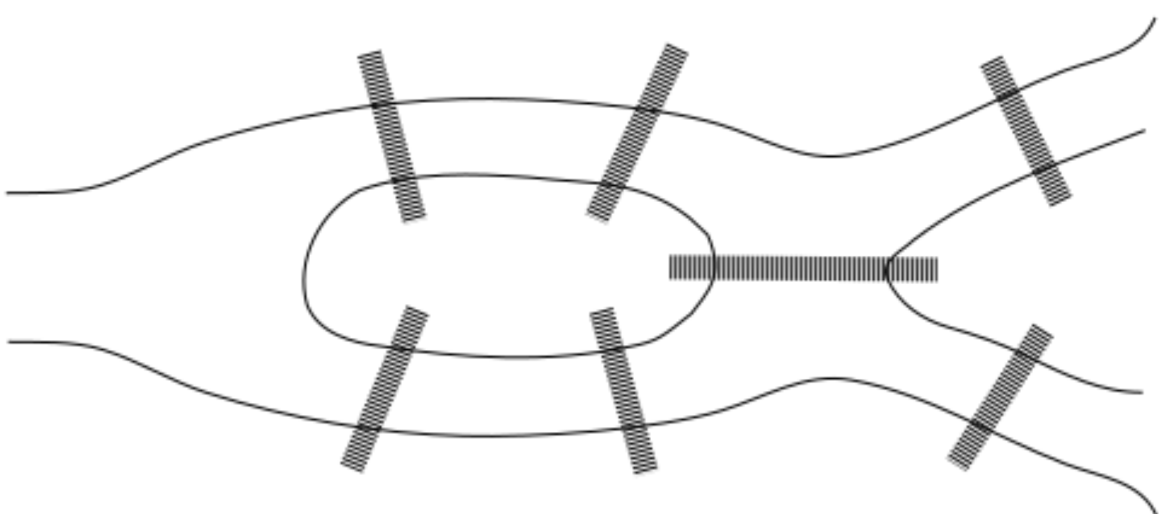
Puzzle of the Week

Parades – 2 – 7 Bridges of Königsberg

This is a map of Königsberg showing the river that runs through it, the island in the middle of the river, and the seven bridges that span the river. The people of Königsberg wanted to have a parade that went over each of their bridges exactly once. They could not seem to find a parade route that visited all seven bridges.



THE CHALLENGE: If you can, find a parade route that crosses each of their bridges exactly once. If you can't, give a reason for why it is impossible.

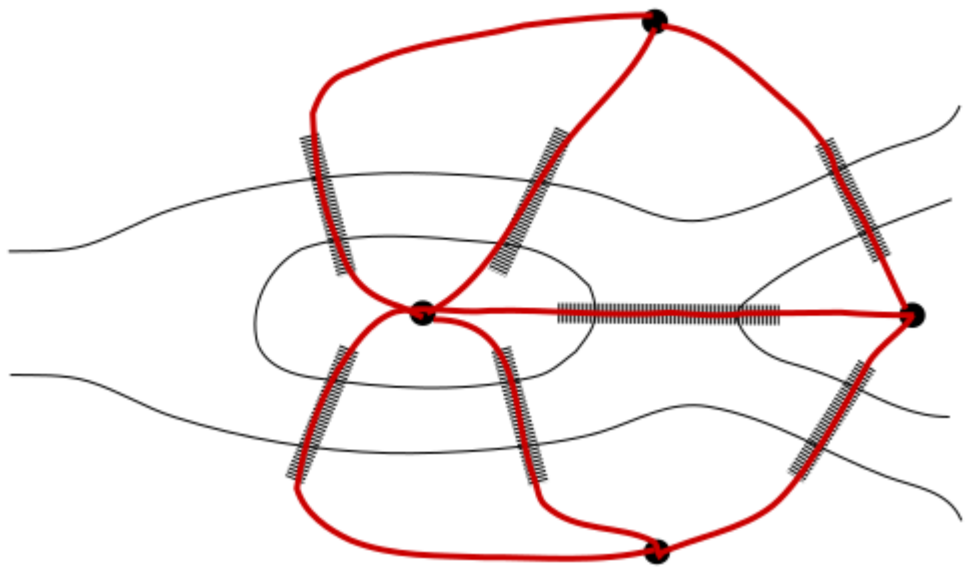


EXPLORATION: If you add another bridge over the river, would that make the problem easier or harder?

Puzzle of the Week

Seven Bridges of Königsberg – Notes

THE CHALLENGE: Start by translating this problem into one that has edges we need to traverse. We saw in the earlier "Parades" Puzzle of the Week that it is crucial to have no more than two nodes where an odd number of sides come in.



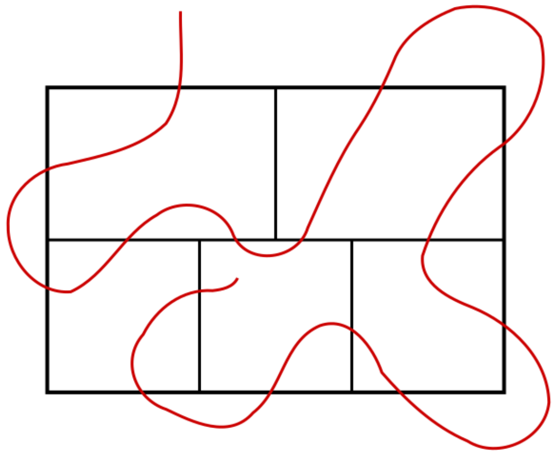
Because all four of the nodes have an odd number of edges coming into them, it will be impossible to find a parade route!

EXPLORATION: If you add one more bridge across the river anywhere, you will make it very easy to solve this problem. That's because two of the four nodes will then have an even number of edges coming into them, and that will leave only two nodes with an odd number of edges coming into them.

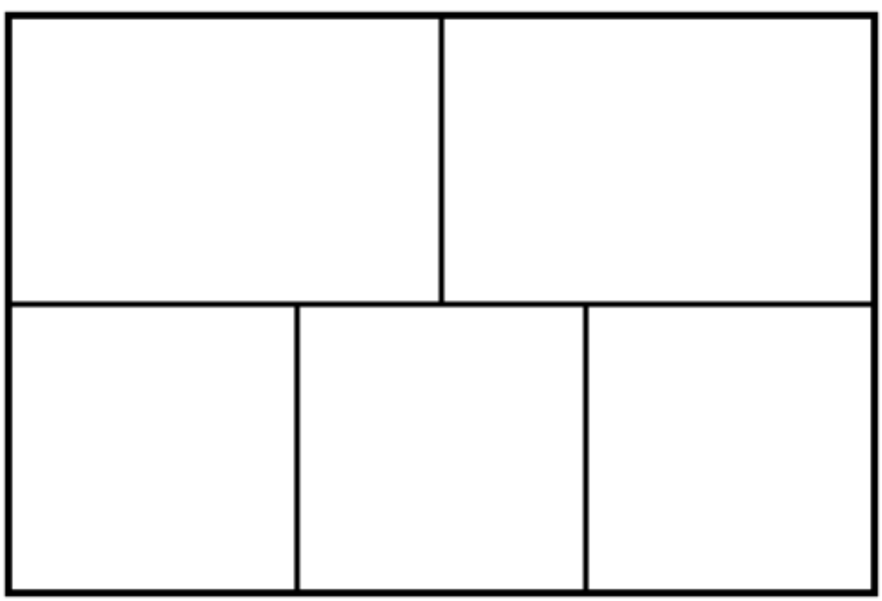
Puzzle of the Week

Parades – 3

The red path is an attempt to find a parade route that crosses each edge of this diagram exactly once. Alas, one of the sides has been missed!



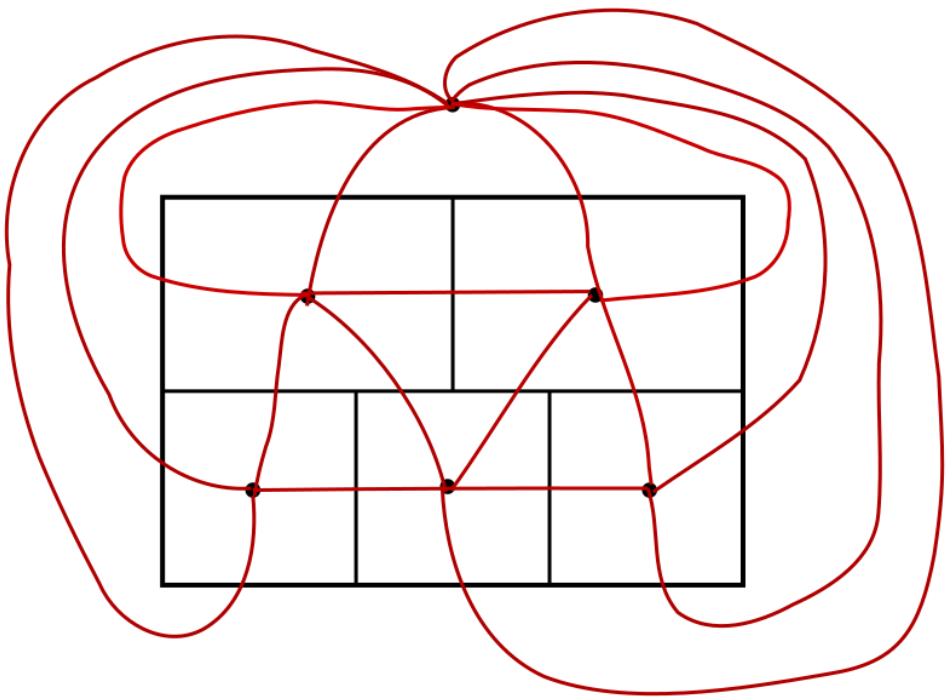
THE CHALLENGE: If you can, find a parade route that crosses each edge of this diagram exactly once. If you can’t, give a reason for why it is impossible.



Puzzle of the Week

Parades – 3 – Notes

THE CHALLENGE: Start by translating this problem into one that has edges we need to traverse. We saw in the earlier "Parades" Puzzle of the Week that it is crucial to have no more than two nodes where an odd number of sides come in.

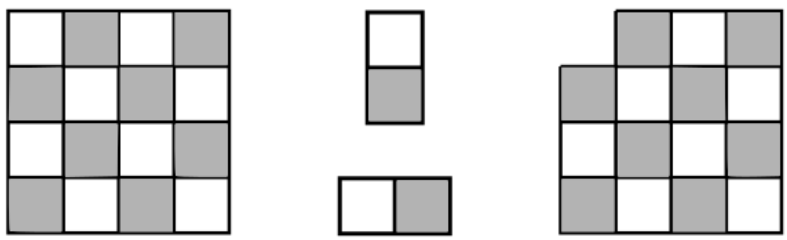


The red lines represent paths that must be taken to cross each of the edges. We are looking for a parade route that will follow each of the red paths once. Unfortunately, there are four nodes that have an odd number of red paths coming into them. Therefore, it is impossible to find a route that will work!

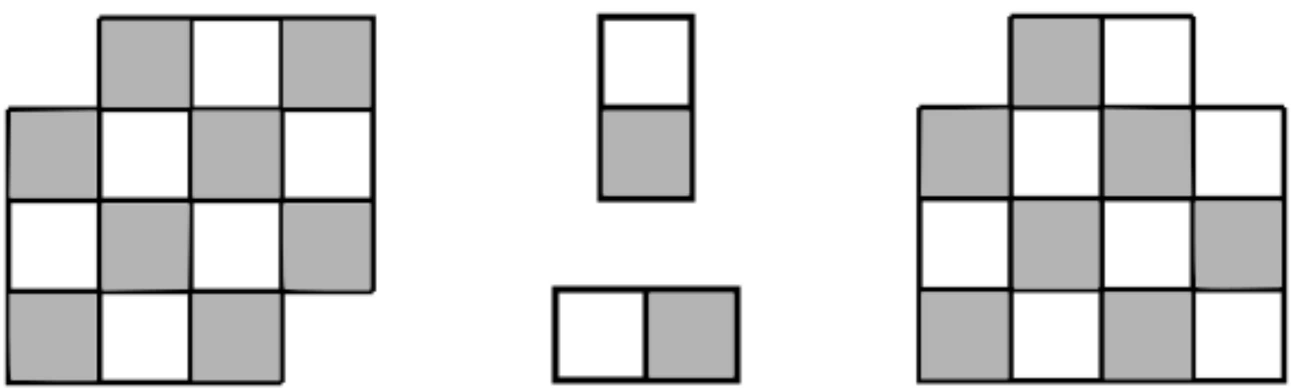
Puzzle of the Week

Dominoes on Checkerboards

The first of these two checkerboards is easy to cover with dominoes. The second one, which has one square missing, is impossible to cover with dominoes.



THE CHALLENGE: Describe why one of these checkerboards is easy to cover with dominoes and the other one is impossible.



EXPLORATION: Explore which pairs of small squares you can remove from the original board and still be able to exactly cover the remaining board with dominoes.

Puzzle of the Week
Dominoes on Checkerboards – Notes

THE CHALLENGE: There are two formulas to look at for this puzzle.

The first is whether the total number of squares available is even or odd. Because a domino has an even number of squares, every time a domino is put on a board, the number of available squares is reduced by an even number. Therefore, if there were an even number of squares available before the domino, it will still be even after the domino, and if there were an odd number of squares before the domino, it will still be odd.

For the introductory examples at the top of the first page, the first example has an even number of squares, so it is numerically possible to end with 0 squares (which is an even number). The second example starts with an odd number of squares, so it is impossible for it to be covered with dominoes.

The second formula is to take the difference between the number of dark squares and the number of light ones. When a domino is put on the board, it covers a dark square and a light square, so the difference of those counts stays the same. For example, the difference is 8 - 6 = 2 before any dominoes are placed (looking at the first challenge board), and it will be 7 - 5 = 2 after placing a domino. A formula such as this, that does not change during moves in a game or puzzle, is called an *invariant*.

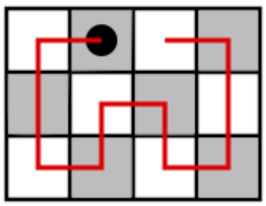
Because the difference starts at 8 - 6 = 2 for the first challenge board, it can never be completely covered by dominoes (which would bring the difference to 0 - 0 = 0).

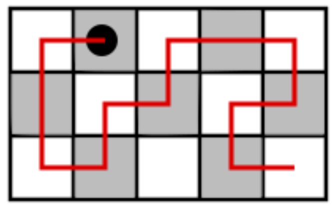
The second board starts with a difference 7 - 7 = 0, so there is a chance that it can be covered. It is actually quite easy to cover the second board with dominoes.

EXPLORATION: Calculating the difference of dark and white squares tells you a lot about which boards can be done. The only difference that gives a chance for success is a difference of 0. If you were to remove four squares, it is possible to have a difference of 0 and not be able to cover the remaining squares - this happens when one of the corner squares gets isolated. However, generally speaking, if the difference is 0 you will almost always be able to cover the board with dominoes.

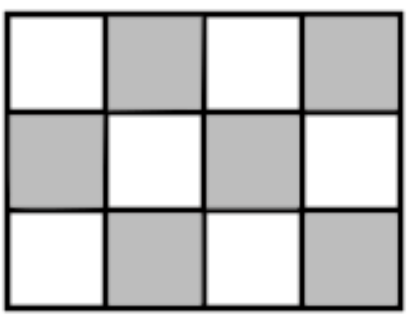
Puzzle of the Week
Paths on Checkerboards

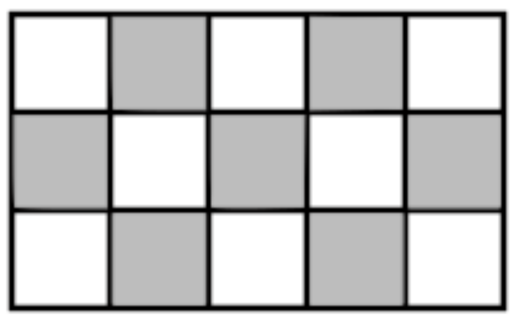
The first 3 by 4 checkerboard has a path that visits every square starting at the black dot. The second 3 by 5 checkerboard has no path that visits every square starting at the black dot.





THE CHALLENGE: For these two checkerboards, identify which starting positions will begin paths that visit every square in the board and which ones do not. What is the difference?





EXPLORATION: Create some other sizes of checkerboards and try various starting positions on these. Do you see any patterns for which starting positions work on each board?

Puzzle of the Week

Paths on Checkerboards – Notes

THE CHALLENGE & EXPLORATION: Trial and error can be quite successful with this puzzle and should be encouraged. With enough time and patience, your students will figure these out. The first challenge board has a complete path starting at any square. The second challenge board has complete paths that work from each of the white squares and none of the dark squares.

That's an interesting result.

The question you want your students to ask at this point is: Why do we see this pattern of results?

Looking at the example paths in the introduction, an important observation is that the color of the squares will alternate along any path. This is because two squares that share a side will be of opposite colors. If you think of a complete path as an alternating list of all the squares of a board, this will be the key to seeing when it is impossible to have a complete path.

For example, look at the 3 by 5 grid in the introduction and in the challenge. It has 8 white squares and 7 black squares. A path that starts at a white square can alternate colors and work: WBWBWBWBWBWBWBW. However, a path that starts at a black square will be stuck: BWBWBWBWBWBWBW without any way of reaching the eighth W square.

For rectangular boards that have an even number of squares and hence the same number of white and black squares, a complete path can start anywhere. For rectangular boards that have an odd number of squares, complete paths must start in the corners or from squares that are the same color as the corners.

The Global Math Project has a lovely video in the Math Without Words section:
https://www.yout-ube.com/watch?v=WC1C394PaYs

Puzzle of the Week

Parity – 3 – Last Number Standing

The numbers 1 to 5 are written on a board. Then pairs of numbers are selected, erased, and replaced by their difference. This continues until there is a single number remaining. In the example below, that number is 1.

1 2 3 4 5 => 1 2 4 2

1 2 4 2 => 2 2 3

2 2 3 => 2 1

2 1 => 1

THE CHALLENGE: How small can that single number be? Can it be 0? Does your answer change if the numbers go from 1 to 6, or from 1 to 7?

1 2 3 4 5 => ?

EXPLORATION: For a given list of numbers, such as 1 to 5, which last numbers are possible? What is the smallest and largest possible last number? When can 0 or the top number occur on the list of possibilities?

Puzzle of the Week

Last Number Standing – Notes

THE CHALLENGE: This is quite similar to the puzzle that asks which numbers are possible when we take the numbers from 1 to 5 and put addition or subtraction signs between each pair of numbers - for example, 1 + 2 + 3 - 4 + 5 or 1 + 2 - 3 + 4 + 5. The advantage of using this form of the puzzle is that no negative numbers will be involved.

A few easy notes to start with. All differences are nonnegative, so the final answer can never be less than 0. The numbers that get created from differences are either with the original numbers or from numbers derived from those numbers. So the maximum number that can be used with a difference is the largest number, which is 5. We want to know which of the numbers from 0 to 5 are possible final answers.

Look at this as an evens and odds problem. Start by counting the number of odd numbers. In the case of going from 1 to 5, that count is 3, which is itself an odd number. So, we have an odd number of odd numbers. Make a list of what happens when you take a difference: 1) If both numbers are even, the result is an even number and there is no change in the total number of odd numbers; 2) if one number is odd and the other is even, the result is an odd number and there is no change in the total number of odd numbers; and 3) if both numbers are odd numbers, then the result is an even number and the total number of odd numbers is reduced by two. In all cases, the total number of odd numbers either stays the same or is reduced by two.

Result: If we start with an odd number of odd numbers, we will end with an odd number (1) of them. If we start with an even number of odd numbers, then we will end with an even number (0) of them.

In the case of going from 1 to 5, we started with an odd number of odd numbers, so the final answer must be odd. The final answer must be 1,3 or 5. Some quick experimenting shows that they are all possible.

The analysis is exactly the same for the numbers from 1 to 6 because there are still three odd numbers.

For the range 1 to 7, there are now an even number of odd numbers, so there will be zero odd numbers at the end and the possible last numbers are 0, 2, 4, or 6 (which can all occur).

EXPLORATION: It is easy to see when 0 is possible. Start by pairing up consecutive numbers from the top and taking their differences. This gives a collection of 1's. If there are an even number of 1's, then you can get a 0, and if there are an odd number, you can get a 1. Not surprisingly, this is the same as finding out whether you started with an even or odd number of odd numbers. Take 1 to 7 as an example: (7 6) (5 4) (3 2) 1 => 1 1 1 1 => (1 1) (1 1) => 0 0 => 0.

You can do the same thing to see if the top number is possible. Pair up consecutive numbers starting at the top leaving out the top number. Take the differences of these pairs. You now have a list of 1's together with the top number. Reduce the list of 1's to either a single 0 or single 1. Take the difference of that with the top number!

Puzzle of the Week

Prisoners with Hats

To celebrate a special occasion, a prison offered to let some prisoners go free if they could pass a test.

The prison had a very large collection of small hats of two colors - say, black and white. On the day of the test, three prisoners were selected and lined up blindfolded facing toward the front of the line. A small colored hat was placed on each prisoner and then the blindfolds were removed. Each prisoner could see all the hats of the prisoners in front of them, but they could not see their own hat or any of the hats behind them.

During the test, the last prisoner in line says a hat color. If that was their hat color, they are set free. Otherwise, they go back to prison. Each prisoner can hear each answer behind them before they give their own answer. No hints to other prisoners were allowed in the tone or manner that each answer was given.

All these rules were made clear to the prisoners the day before the test, and they were allowed to strategize together.

THE CHALLENGE: Describe a strategy that guarantees the largest number of prisoners will go free.

EXPLORATION: How does your solution change if there are four or even five prisoners? Devise a strategy that works for any number of prisoners.

Puzzle of the Week

Prisoners with Hats – Notes

THE CHALLENGE: Your students will be tempted to come up with schemes that involve cheating by having the color spoken in some sort of unusual way that gives a clue to the other prisoners. This of course is not allowed. They may also be tempted to think if both hats in front of them are the same color, then their hat is likely to be the other color - it is important to point out that there is a very large supply of hats so that there will be no bias regarding running out of one type of hat.

The other thing they will be tempted to do is to rely on good luck. This puzzle specifically looks for guarantees of success and not ways of being lucky.

One common strategy suggestion is to have the first prisoner say the hat color of the person directly in front of them. Unfortunately, that strategy only guarantees that one person will go free.

The best strategy is for the first prisoner to do the following. They count the number of black hats in front of them. If the number of black hats is even, they say “Black,” and if it is odd, they say “White.” This strategy does not guarantee that the first prisoner will go free, but there is no way to guarantee that.

After the first prisoner’s turn, the second prisoner can figure out their hat. Suppose the first prisoner said “Black.” Then the second prisoner knows that their hat and the third prisoner’s hat are either both black or both white - by looking at the hat on the third prisoner, it is easy to know which case it is. Similarly, if the first prisoner says “white,” then the two remaining hats must be opposite colors.

Finally, the third prisoner has heard the previous two hat announcements. If the first one said “Black,” then the two hats must be the same color and their hat must be the same as the second person’s hat. If the first one said “White,” then the two hats must be different colors and their hat is the opposite of the second person’s hat.

This strategy two prisoners will go free. If they're lucky, the third prisoner will go free as well.

EXPLORATION: For more than three prisoners, the strategy starts off exactly the same.way. Note that if the original strategy is put in terms of the hats being the same or different (instead of using even and odd), then the jump to more than three prisoners requires a major insight. The best way to explain how this will work is to have your students put on pretend hats and act out how they can figure out their hat color using ideas about even and odd numbers.

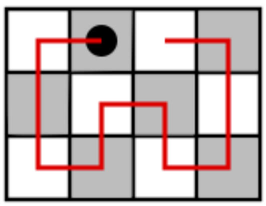
TEDEd has a lovely video for the prisoner hat riddle:

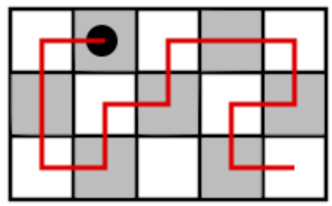
https://www.youtube.com/watch?v=N5vJSNXPEwA

Puzzle of the Week

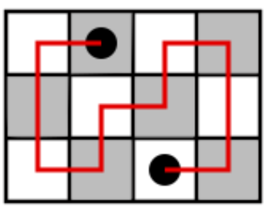
Paths on Checkerboards – 2

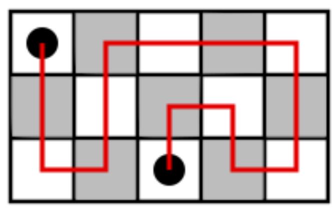
In the Puzzle of the Week “Paths on Checkerboards - 1” we looked at when it was possible, starting at a given point, to make a path on a checkerboard that visited every square. Starting at the black dot, the first 3 by 4 checkerboard has a path and the second 3 by 5 checkerboard does not.



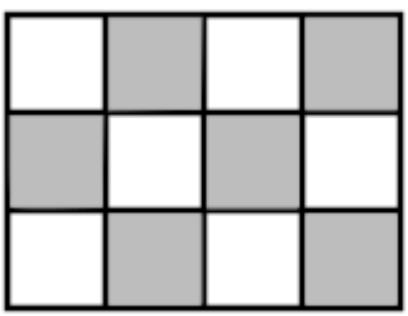


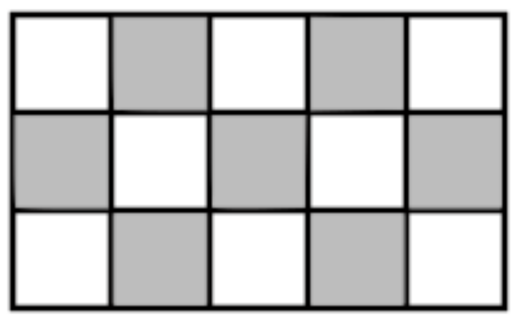
For this puzzle we have both a starting and ending point, and ask the question of whether there is a path that starts at one point and visits every point exactly once on the way to ending at the other point.





THE CHALLENGE: For these two checkerboards, identify which pairs of starting and ending positions have a path that links them that visits every square in the board once, and which ones do not. What is the difference?





EXPLORATION: Create some other sizes of checkerboards and try various starting positions on these. Do you see any patterns for which starting positions work on each board?

Puzzle of the Week

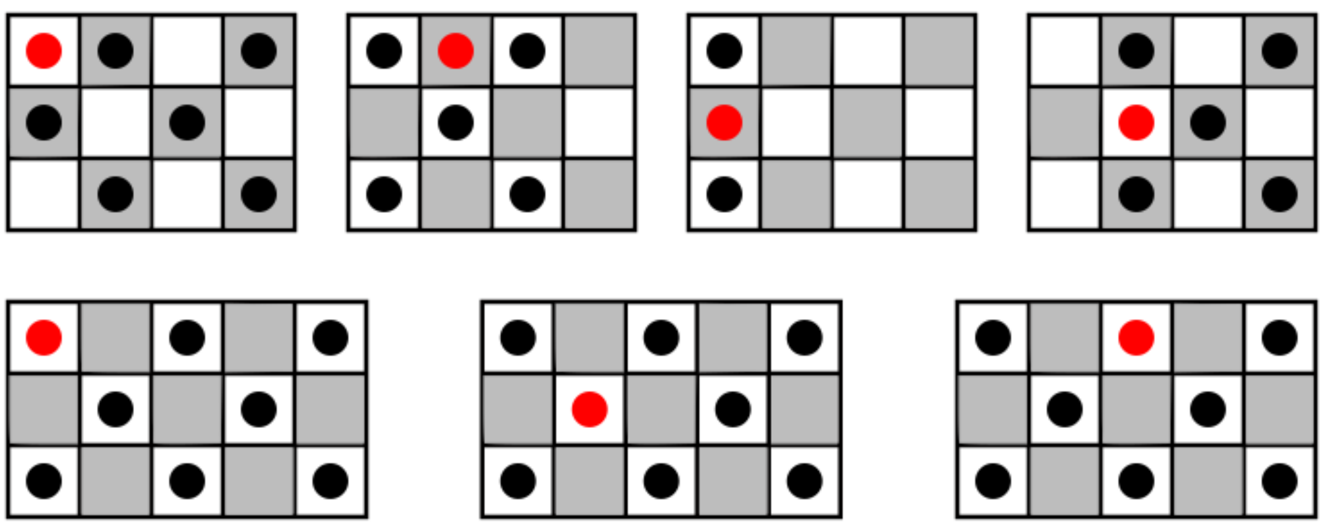
Paths on Checkerboards – 2 – Notes

THE CHALLENGE & EXPLORATION: In "Paths on Checkerboards - 1" we saw that the sequence of colors on a path will alternate as one moves along the path.

As we saw in that earlier puzzle, for checkerboards with an even number of squares, such as the 3 by 4 board, we must end on the opposite color from what we started on. For checkerboards with an odd number of squares, such as the 3 by 5 board, we must begin and end on the color that is in one of the corners of the board.

That leaves the question - If we follow those rules, is that enough?

One way to analyze this question is to set the position of one of the points and see what all the endpoints are for good paths. We can exploit mirror symmetry (flipping around the middle of the board in one direction or the other) to reduce the number of possible cases to look at. Here is what we get for the two boards. The red dot is a fixed starting point, and the black dots are places with successful endpoints.



As you can see, most positions work. For the ones that fail, it is not a matter of having trouble with alternating colors, it is a problem with getting boxed into corners. This confinement problem is even more troublesome for boards with only two rows. For four or more rows, I believe the confinement problems go away.

Puzzle of the Week

Pirates with Gold – 1

There are three pirates on an island, and they have a total of 12 gold coins.

The rules are:

1. The pirates are very smart.
2. Each pirate wants as much of the gold as possible for themselves, and does not care about the others.
3. The youngest pirate must propose a plan for splitting up the gold. If the plan is acceptable to over half of all the pirates (including the youngest), then the plan is adopted. Otherwise, the youngest pirate is forced to leave the island with no gold, and the new youngest pirate must propose a plan.

THE CHALLENGE: What is the most gold coins the youngest pirate can get in an acceptable plan?

Puzzle of the Week

Pirates with Gold – 1 – Notes

THE CHALLENGE: This puzzle is a wonderful example of using two related problem-solving techniques - learning from examples and learning from simpler versions of the problem.

Let’s work our way up to three pirates.

1 Pirate: This is easy. The youngest pirate gets all 12 gold coins.

2 Pirates: The youngest pirate gets nothing. If the youngest pirate tried to have any of the gold, the other pirate would vote against the plan, forcing the youngest to leave and thereby getting everything anyway.

3 Pirates: The youngest pirate need only convince one other pirate to vote for the plan. As we saw in the two-pirate case, the second-youngest pirate will get nothing if the initial plan is rejected. So, the youngest pirate’s plan just needs to give the second-youngest pirate 1 gold coin to give the second-youngest pirate a reason to vote for the plan.

Giving away 1 gold coin means the youngest pirate gets to keep 11 gold coins!

In “Pirates with Gold - 2” we will see what happens when there even more pirates.

Puzzle of the Week

Pirates with Gold – 2

There are five pirates on an island, and they have a total of 12 gold coins.

The rules are:

1. The pirates are very smart.
2. Each pirate wants as much of the gold as possible for themselves, and does not care about the others.
3. The youngest pirate must propose a plan for splitting up the gold. If the plan is acceptable to over half of all the pirates (including the youngest), then the plan is adopted. Otherwise, the youngest pirate is forced to leave the island with no gold, and the new youngest pirate must propose a plan.

THE CHALLENGE: What is the most gold coins the youngest pirate can get in an acceptable plan?

EXPLORATION: Explore what happens when there are more than five pirates.

Puzzle of the Week

Pirates with Gold – 2 – Notes

THE CHALLENGE: This puzzle provides great practice with two problem-solving techniques. The first is to learn from examples and simpler versions of the problem. The second is to use a table or other organizational scheme to arrange the data in an easy to understand way.

In “Pirates with Gold – 1” we looked at what happens when there are 1, 2, or 3 pirates. Because there are starting to be a lot of cases to look at, we’ll organize the information into a table. The table will show how many gold coins the best plan (for the youngest pirate) will give to each of the pirates. For ease of reference, we’ll name the pirates A, B, C, D, and E, in order of age, with A the youngest and E the oldest.

An X in the table means the pirate was not involved in that version of the problem.

E	D	C	B	A
12	X	X	X	X
12	0	X	X	X
0	1	11	X	X
1	2	0	9	X
2	0	1	0	9

We looked at the plans for the first three rows in “Pirates with Gold - 1.” To review, if E is the only pirate, then E gets all the gold. If D and E are the only pirates, then D must give all the gold to E or E won’t vote for D’s plan. If the pirates are C, D, and E, then C needs one more vote - C can get that vote by giving D more gold than D would get if D votes against C’s plan.

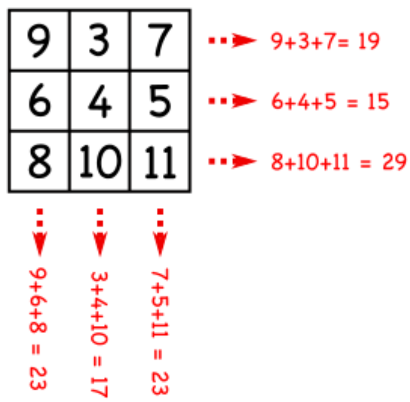
Suppose the pirates are B, C, D, and E. B needs two other votes to get B’s plan approved. That means B must give more gold to two pirates than they would get if they reject B’s plan. The best way to do that is for B to give E one piece of gold and D two pieces of gold.

Finally, suppose all five pirates are involved. A needs two other votes to get A’s plan approved. Once again, that means A must give more gold to two pirates than they would get if they reject A’s plan. This is mostly easily done by giving E two pieces of gold and C one piece of gold.

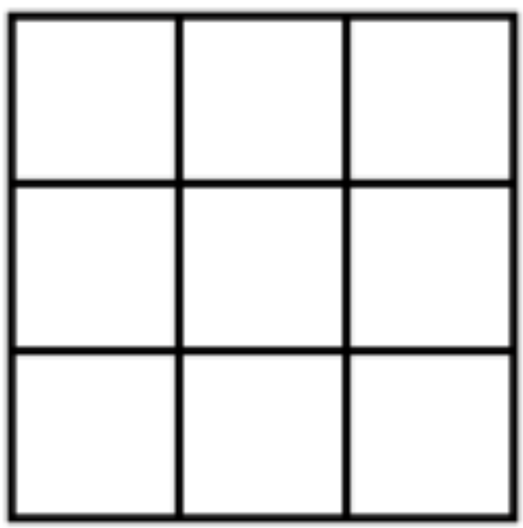
The answer to the puzzle is: The youngest pirate gets to keep 9 pieces of gold!

Puzzle of the Week
Prime Squares

The first few prime numbers are: 2, 3, 5, 7, 11, 13, 17, 19, and 23. A *Prime Square* is a square grid of numbers in which the numbers in each row and column add up to a prime number. This 3 by 3 square, which uses the numbers from 3 to 11, is almost a Prime Square - it fails because the second row adds up to 15, which is not a prime!



THE CHALLENGE: Use the numbers from 1 to 9 to form a 3 by 3 Prime Square.



1 2 3 4 5 6 7 8 9

Puzzle of the Week

Prime Squares – Notes

THE CHALLENGE: A good way to warm up for this puzzle is to start with the Odd Squares Puzzle. Because the only even prime number is too small to be the sum of three positive numbers, we know all the sums for this puzzle will be odd numbers. As mentioned in the Notes section of that puzzle, for a row and column to add up to an odd number, it must have 1 or 3 odd numbers in it. Because there are a total of 5 odd numbers, that means one row will have 3 odd numbers and two rows will have one odd number. Similarly, one column will have 3 odd numbers and two columns will have one odd number.

ODD	ODD	ODD
EVEN	EVEN	ODD
EVEN	EVEN	ODD

We can slide the rows and columns around and not really change the answer in an interesting way, so we might as well assume that the answer looks like this:

The challenge now is to pick numbers that will add up to primes. There are lots of ways to do this, perhaps too many!

Things can be slid around without creating an answer that is really different, so we can assume that 2 is in the upper left corner of even numbers, and the upper right corner is less than the lower left corner.

ODD	ODD	ODD
2	4	ODD
6	8	ODD

ODD	ODD	ODD
2	4	ODD
8	6	ODD

ODD	ODD	ODD
2	6	ODD
8	4	ODD

Here are the answers with (2 4) (8 6) for the two rows of even numbers::

1	7	3
2	4	5
8	6	9

1	7	9
2	4	5
8	6	3

3	9	1
2	4	7
8	6	5

3	9	7
2	4	1
8	6	5

7	1	3
2	4	5
8	6	9

7	1	9
2	4	5
8	6	3

9	3	1
2	4	7
8	6	5

9	3	7
2	4	1
8	6	5

Puzzle of the Week

Self-Describing Numbers – 1

The number 1210 is a *Self-Describing Number* because each digit in order describes how many digits of that type there are - there is 1 0, 2 1's, 1 2, and 0 3's. SImilarly, 2020 is a Self-Describing Number because it has 2 0's, 0 1's, 2 2's, and 0 3's.

1	2	1	0
# of 0's	# of 1's	# of 2's	# of 3's

2	0	2	0
# of 0's	# of 1's	# of 2's	# of 3's

THE CHALLENGE: Find a Self-Describing Number that has five digits.

# of 0's	# of 1's	# of 2's	# of 3's	# of 4's

EXPLORATION: Why are there no Self-Describing Numbers with 1, 2, or 3 digits?

Puzzle of the Week

Self-Describing Numbers – 1 – Notes

THE CHALLENGE: A playful, disorganized approach to this is fine and should be encouraged.

Here are a few analytical ideas for attacking this puzzle.

Result 1: The sum of the digits is the number of digits. The digits count the digits of each type, so the sum of the digits counts all the digits.

Result 2: The sum of the products is the number of digits. Put another way, if you take the sum of multiplying each digit by the size of the digits it is counting, that sum must be the number of digits. This result follows from Result 1 because the sum of the products counts up all the digits involved in the number.

Look at the two given examples. In 1210, the sum of the products is (1 x 0) + (2 x 1) + (1 x 2) + (0 x 3) = 4. In 2020, the sum is (2 x 0) + (0 x 1) + (2 x 2) + (0 x 3) = 4.

Result 3: The rightmost, low-order digit, the ones digit, is 0. From Result 2, this digit must be 0 or 1 (or the product would be too large). Suppose it is 1. To avoid using variables, let's suppose we have a 6-digit number. Then the ones place counts the number of 5's. If there is a 5 someplace, then, by result 1, the only other non-zero digit must be a 1. But then we would have four 0's, which is impossible.

Result 4: The high-order digit is not 0. This must be true for this number to be considered a number.

Okay, let's construct some answers for 5-digit numbers.

There is a basic tension between results 1 and 2. To keep those sums the same, there must be an increase in 0's. For example, for each 2 there must be an extra 0, for each 3 there must be two extra 0's, for each 4 there must be three extra 0's, and so on. That's why most self-describing numbers have a lot of 0's.

If the number of 0's were 1, there would need to be nonzero numbers in every place except the far right place. But having all those nonzero numbers would necessitate having more 0's to make Result 2 work out. So, having only one zero in numbers with more than four digits is impossible. We can call that Result 5 if you like.

If there are 2 0's, the number is either 22100 or 21200. Of those, only 21200 works, and **that is the answer!**

If there are 3 0's, the number would be 32000, which doesn't work.

EXPLORATION: Let's look at the cases individually. Remember the four results when looking at these.

1 digit: Results 3 and 4 make this impossible.

2 digit: The number would have to be 20, which is not self describing.

3 digit: The number would have to be either 300, 210, or 120. None of these is self describing.

Puzzle of the Week

Self-Describing Numbers – 2

The number 1210 is a *Self-Describing Number* because each digit in order describes how many digits of that type there are - there is 1 0, 2 1’s, 1 2, and 0 3’s. SImilarly, 2020 is a Self-Describing Number because it has 2 0’s, 0 1’s, 2 2’s, and 0 3’s.

1	2	1	0
# of 0's	# of 1's	# of 2's	# of 3's

2	0	2	0
# of 0's	# of 1's	# of 2's	# of 3's

THE CHALLENGE: Find a Self-Describing Number that has seven digits.

# of 0's	# of 1's	# of 2's	# of 3's	# of 4's	# of 5's	# of 6's

EXPLORATION: Why are there no Self-Describing Numbers with 6 digits?

Puzzle of the Week
Self-Describing Numbers – 2 – Notes

THE CHALLENGE & EXPLORATION: A playful, disorganized approach to this is fine and should be encouraged.

Here are some useful results from the Notes on Self-Describing Numbers – 1:

Result 1: The sum of the digits is the number of digits.

Result 2: The sum of the products is the number of digits.

Result 3: The rightmost, low-order digit, the ones digit, is 0.

Result 4 & 5: The high-order digit is at least 2 for numbers with at least five digits.

There is a basic tension between results 1 and 2. To keep those sums the same, there must be an increase in 0's. For example, for each 2 there must be an extra 0, for each 3 there must be two extra 0's, for each 4 there must be three extra 0's, and so on. That's why most self-describing numbers have a lot of 0's.

Suppose the high-order digit is 2, so there are only two 0's. For a five-digit number, that means there are three nonzero digits. In Self-Describing Numbers – 1 we found that the only solution was 21200. For a six-digit number, there would be four nonzero digits. Also, since the digits have to add up to six, that would force the number to have two 1's and two 2's. None of the ways of mixing two 1's and two 2's work. The same thing keeps happening for longer numbers that have two 0's. This gives:

Result 6: The high order digit is at least 3 for numbers with at least six digits.

Let's keep playing with the number of 0's. What would be too many? Suppose the number of 0's is within three of the total number of digits. That would leave at most two more nonzero digits. One of those would be a 1 for the large 0 count. The only other possible nonzero entry would have to be for the number of 1's, which is not going to work out. We get yet another result:

Result 7: For numbers with at least six digits, the number of nonzero entries must be more than 3.

If you combine results 6 and 7, you will see that there cannot be any solutions for six-digit numbers. It is surprising that this works for Self-Describing numbers that have 5 or 7 digits, but not for ones that have 6 digits.

For 7-digit numbers, results 6 and 7 together guarantee that, if there is a solution, the number of 0's is 3. The other entries in the number must total 4. A few quick experiments quickly leads to the answer.

The answer is 3211000!

Puzzle of the Week

Self-Describing Numbers – 3

The number 1210 is a *Self-Describing Number* because each digit in order describes how many digits of that type there are - there is 1 0, 2 1's, 1 2, and 0 3's. SImilarly, 2020 is a Self-Describing Number because it has 2 0's, 0 1's, 2 2's, and 0 3's.

1	2	1	0
# of 0's	# of 1's	# of 2's	# of 3's

2	0	2	0
# of 0's	# of 1's	# of 2's	# of 3's

THE CHALLENGE: Find a Self-Describing Number that has ten digits.

# of 0's	# of 1's	# of 2's	# of 3's	# of 4's	# of 5's	# of 6's	# of 7's	# of 8's	# of 9's

EXPLORATION: Do you see a pattern in your answer for ten digits that will help you find other examples with a large number of digits?

Puzzle of the Week

Self-Describing Numbers – 3 – Notes

THE CHALLENGE: A playful, disorganized approach to this is fine and should be encouraged.

Here are some useful results from the Notes on Self-Describing Numbers – 1 and – 2:

Result 1: The sum of the digits is the number of digits.

Result 2: The sum of the products is the number of digits.

Result 3: The rightmost, low-order digit, the ones digit, is 0.

Result 4 & 5: The high-order digit is at least 2 for numbers with at least five digits.

Result 6: The high-order digit is at least 3 for numbers with at least six digits.

Result 7: For numbers with at least six digits, the number of nonzero entries must be more than 3.

It is probably helpful to have the answer for 7 in front of us: 3211000. Looking at this, it is tempting to think that the number of nonzero entries is four for numbers with at least six digits. From Result 7, we know that the number of nonzero entries cannot be less than four. Can it be more than four?

We immediately know two of the nonzero entries - the high order digit that gives the number of 0's (which is a number that is at least 3), and the place that corresponds to that digit - if that place has a value greater than 1, then that causes there to be that many 0's and that many of some other number, and things quickly spiral out of control. So, there is at least one 1. That gives us three nonzero entries. It is now impossible to have a single 1 (how would we fill in the number of 1's), so the number of 1's must be at least two. So we get an additional 1 corresponding to the place that has the number of 1's.

And that's it. Any additional entries would once again cause a snowball effect that would cause the numbers to get too big. At last, we arrive at the answer.

The answer for 10 is 6210001000!

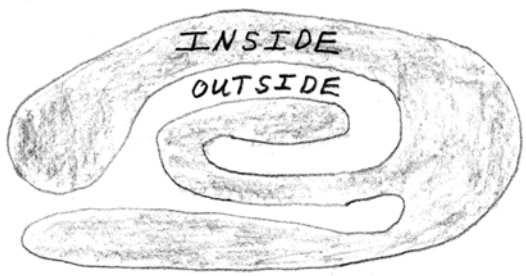
EXPLORATION: The answer for 7 was 3211000 and the answer for 10 was 6210001000. The pattern X210….01000 looks promising. Let's look at how we can apply it.

It looks like, as we go up from 7, we can increase the number of 0's by 1 each time, and put that new 0 between the two 1's. If you check it out, it always works. Well, it works until we get to 13 with 9210000001000. After 13, the first digit becomes too large to be a single digit, but it was fun while it lasted.

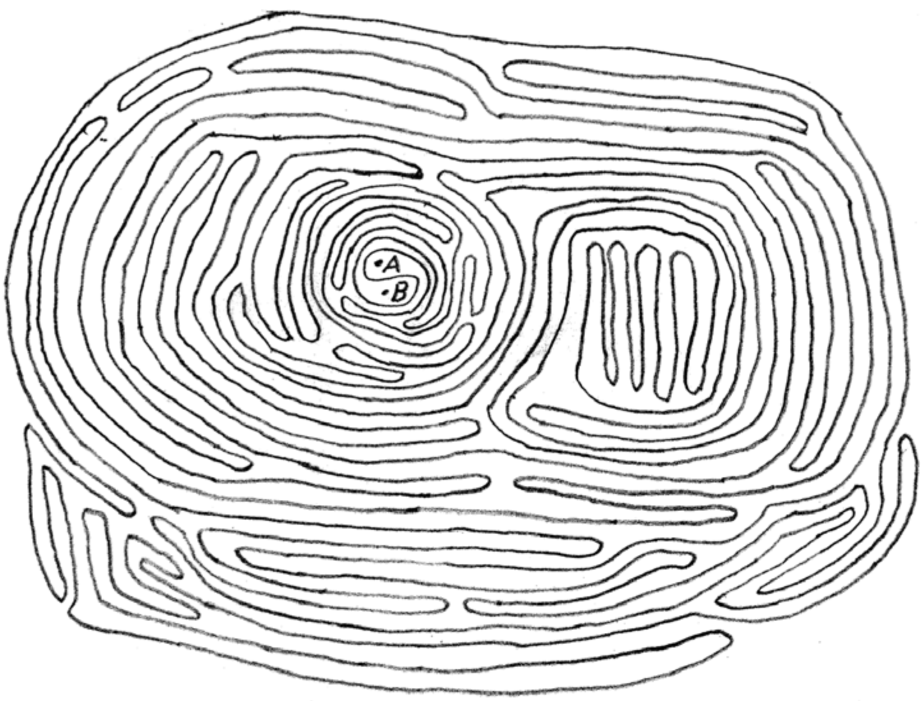
Puzzle of the Week

Insides and Outsides

A *simple closed curve* is a single continuous line that does not bump into itself and has no breaks. An important characteristic of these curves is that they have an inside and an outside.



THE CHALLENGE: For points A and B, determine which is on the inside of the curve and which is outside.



EXPLORATION: Rather than tracing around this entire closed curve, can you find a simple way to test whether a point is on the inside or outside of the curve?

Puzzle of the Week

Insides and Outsides – Notes

THE CHALLENGE & EXPLORATION: Rather than tortuously following the paths around the curve, the key is to allow yourself to cross the curve. Each time you cross the curve you will either go from the inside to the outside, or from the outside to the inside.

Hence, to decide whether these points are inside or outside, start from the outside and count how many times you must cross the curve to get to one of the points. If it is an even number of times, then the point is on the outside. If it is an odd number of times, then the point is on the inside.

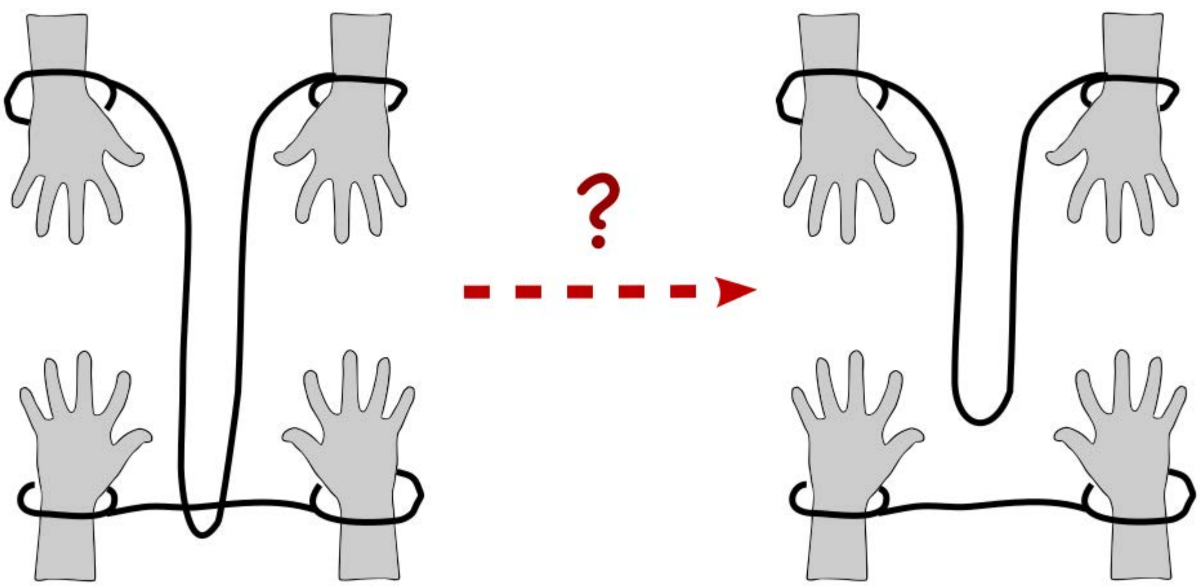
Counting crossings, A is on the Outside and B is on the inside.

You will notice that, no matter which path you take from the outside to get to A, it will always take an even number of crossings to get there.

Puzzle of the Week

Are Their Hands Tied?

THE CHALLENGE: There are two people. Each person ties their hands together (loosely) with a string. As this is being done, the strings loop inside each other as shown. How do the two people get apart without untying their strings?



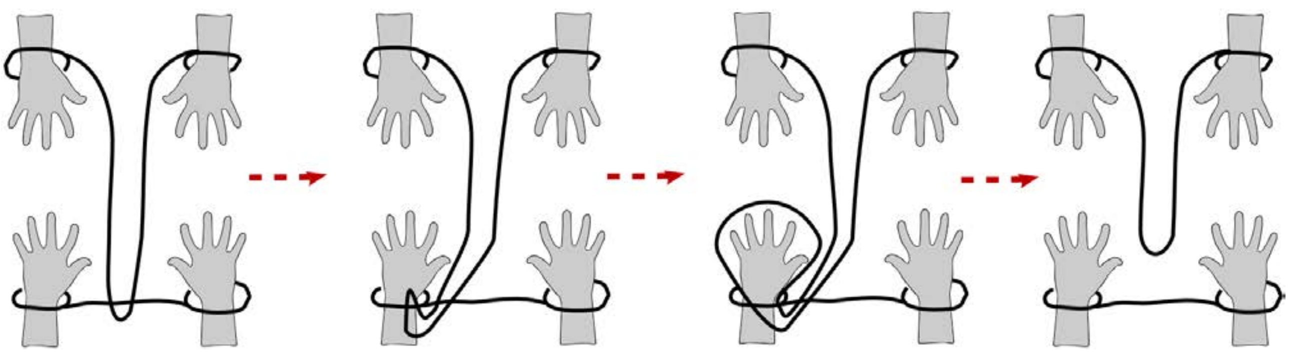
Puzzle of the Week

Are Their Hands Tied? – Notes

THE CHALLENGE: The key insight is to let go of the idea that the string is tied tightly around the wrists. The best way to visualize what to do is to open up the string around one of the wrists and fully withdraw that hand from the string. Once this is done, all you have to do is slide the other person's string off.

That feels like cheating, but it isn't! Topologically, a little bit of space is the same as a great deal of space. In any event, you don't need to withdraw your hand to solve the puzzle. However, withdrawing the hand will help you see why the solution works.

To solve this without pulling a hand out, take the loop that goes around the other string and move it inside one of the loops around a hand - then move it around that hand and out the other side. You should find that the two strings are no longer entangled!

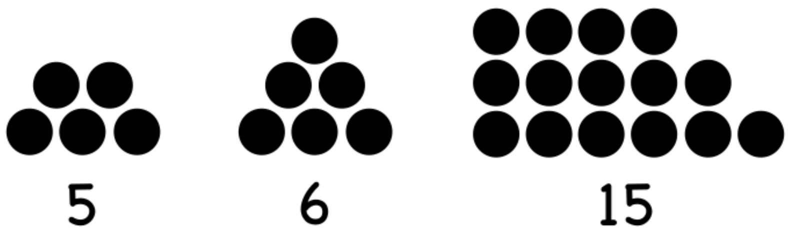


Reading this is a lot harder than trying it out for yourself now that you've got the idea. Grab some string and a partner and try it out!

Puzzle of the Week

Trapezoidal Numbers – 1

Trapezoidal Numbers are the sum of two or more consecutive numbers. They deserve their name because you can make a trapezoid with that many dots, as pictured in the examples below. Note that having 1 dot on the top row is stretching the idea of being a trapezoid a bit, but it is allowed for these numbers.



THE CHALLENGE: Show that every odd number is a Trapezoidal Number.

EXPLORATION: Can you find some even numbers that are Trapezoidal Numbers? Can you find some that aren't?

Puzzle of the Week

Trapezoidal Numbers – 1 – Notes

THE CHALLENGE: Every odd number is of the form 2n + 1. Because 2n + 1 = n + (n + 1), that shows every odd number, larger than 1, is a Trapezoidal number..

EXPLORATION: Among the first few even numbers, we have the following:

- 2: Not trapezoidal
- 4: Not trapezoidal
- 6: 6 = 1 + 2 + 3
- 8: Not trapezoidal
- 10. 10 = 1 + 2 + 3 + 4
- 12: 12 = 3 + 4 + 5
- 14: 14 = 2 + 3 + 4 + 5
- 16: Not trapezoidal

It looks like powers of 2 are never trapezoidal. Will follow that up in a later Puzzle of the Week.